What is and what could be Nonabelian Algebraic Topology?

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Also 17 or more Bangor research students: many have contributed key ideas and all have been essential for progress.

Why change current algebraic topology?

Not so good on involving the fundamental group and its actions. Transition from homology to homotopy is difficult to understand, and seems contrived.

'Local-to-global problems' In 1957 in Oxford and I learned from Dick Swan of the importance of these problems in mathematics, and how they were then tackled by sheaves and spectral sequences.

Here is a nonabelian local-to-global situation.

Knot trick! Rule at each crossing:

$$z = xyx^{-1}.$$

Put them together for the pentoil to get

$$xyxyxy^{-1}x^{-1}y^{-1}x^{-1}y^{-1} = 1$$

We introduce new 'higher dimensional nonabelian methods for local-to-global problems' which, when they work, give precise and powerful information.

More information: do a web search on

higher dimensional algebra, or

higher dimensional group theory

(over 17,000 page views since May, 2000.)

One main motivation: aesthetic.

To find algebra which reflects the geometry, not force the geometry into a given algebraic mode.

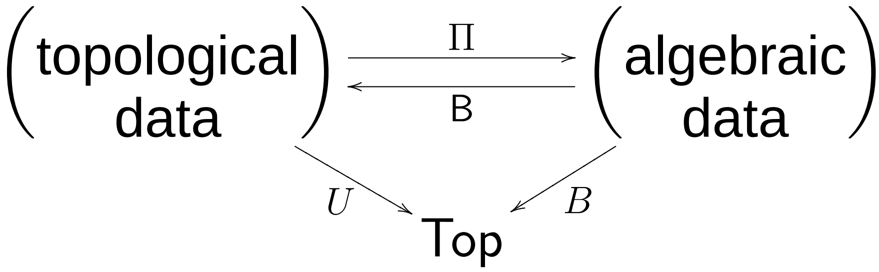
It turns out that this allows for more powerful theorems and new calculations.

In practice, this means that if groupoids seem to arise naturally, use them!

Re-express original intuitions of ‘composing’ pieces!

WHAT IS NONABELIAN ALGEBRAIC TOPOLOGY?

ONE THEME:



1) $\Pi \circ \mathsf{B} \simeq 1$

2) U is a forgetful functor and $B = U \circ \mathsf{B}$

3) Π satisfies a van Kampen theorem (it preserves certain colimits)

4) $\exists$ natural transformation $1 \to \mathsf{B}\Pi$ with 'good' properties

5) ideally: homotopy classification:

$$[UX_*, BC] \cong [\Pi X_*, C]$$

What type of algebraic data?

Higher dimensional algebra is concerned with algebraic structures whose operations are defined under geometric conditions!

(My definition!)

The ones we want allow

algebraic inverse to subdivision!

Hence good for local-to-global!

Part of the algebra we use is:

$$(\text{groups}) \subseteq (\text{groupoids}) \subseteq (\text{multiple groupoids})$$

Higher categorical methods

Two big words:

WHAT IF?

Key construction:

Space X_∞ and an increasing sequence of subspaces

$$X_* : X_0 \subseteq X_1 \subseteq X_2 \subseteq \cdots \subseteq X_\infty .$$

With the obvious morphisms, this gives the category FTop.

Example: The n-cube $I^n = I \times \cdots \times I$ with skeletal filtration I^n_*.

So we can form the filtered singular cubical complex $R^\square(X_*)$ which in dimension n is

$$R^\square(X_*)_n = \mathsf{FTop}(I^n_*, X_*).$$

This is a

cubical set with compositions, and connections.

Why work cubically? In that setting we can describe:

Algebraic inverse to subdivision;

Commutative cubes.

So we define

$$\varrho_n^{\square}(X_*) = (R_n^{\square}(X_*))/\equiv$$

where $\equiv$ is

homotopy through filtered maps rel vertices

(i.e. keep the vertices fixed in the homotopy).

$$p : R^{\square}(X_*) \to \varrho^{\square}(X_*)$$

Major results:

1) All the above structure is inherited by $\varrho^{\square}(X_*)$ to make it a cubical ω-groupoid with connections.

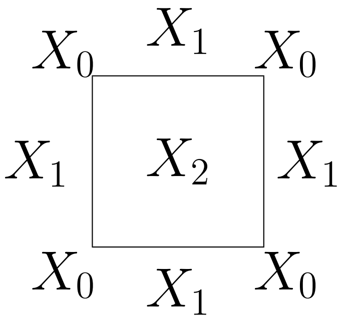
A higher homotopy groupoid!

2) p is a Kan fibration of cubical sets.

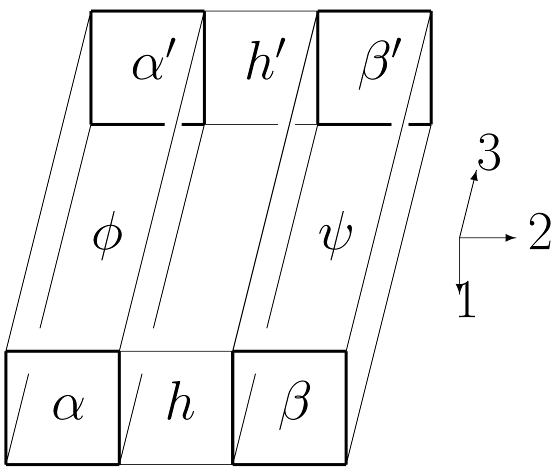
3) $\varrho^{\square}$ preserves certain colimits (van Kampen!)

The last was the aim of the programme started in 1966. All these are due to Higgins and RB, 1981, JPAA.

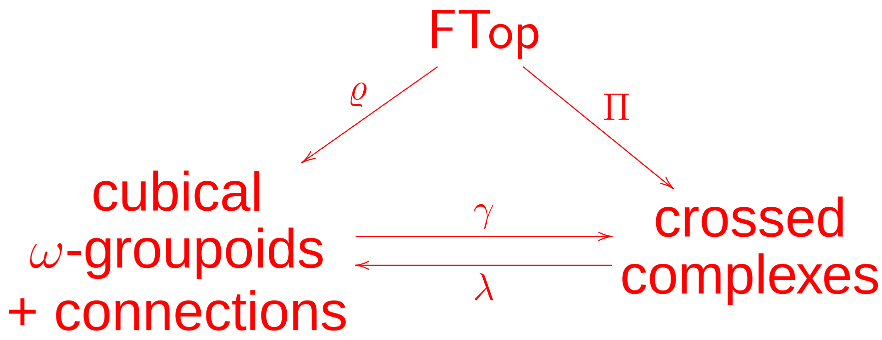
All proofs are tricky in dimensions > 1. Here is a picture for the elements of $R_2^{\square}(X_*)$.



Here is a picture for the proof that composition $\circ_2$ is well defined in $\varrho_2^{\square}(X_*)$.



Major diagram:



- $\lambda, \ \gamma$ are inverse adjoint equivalences
- of monoidal closed categories,
- $\varrho, \ \Pi$ are homotopical functors,
- which preserve certain colimits
- and certain tensor products
- $\gamma\varrho \simeq \Pi$

Also there exists an adjoint pair of functors

$$\text{crossed complexes} \underset{\Phi}{\overset{\Delta}{\rightleftarrows}} \text{chain complexes with a groupoid of operators}$$

with Δ left adjoint to Φ.

This relates crossed complexes to standard chain complex theory.

Taster result: generalise the relative Hurewicz Theorem to a Homotopical Excision Theorem (which led to an n-adic Hurewicz Theorem with Loday.)

These are not accessible by traditional algebraic topology.

Follow from Generalised van Kampen Theorems.

Homotopical Excision

View the excision map of open subsets as part of a pushout of pairs of (pointed) spaces

$$\begin{array}{ccc} (X\cap Y, X\cap Y) & \longrightarrow & (Y,Y) \\ \downarrow & & \downarrow \\ (X, X\cap Y) & \longrightarrow & (X\cup Y, Y) \end{array}$$

Consider the diagram

$$\begin{array}{ccc} \pi_n(X, X\cap Y) & \longrightarrow & \pi_n(X\cup Y, Y) \\ \partial \downarrow & & \downarrow \partial \\ \pi_1(X\cap Y) & \longrightarrow & \pi_1(Y) \end{array}$$

where $\partial = 0$ if $n \geqslant 3$. The top relative groups are modules over the bottom groups
(crossed modules if $n = 2$).

Homotopical Excision [Brown-Higgins, 1981]

Suppose $(X, X \cap Y)$ is $(n-1)$-connected, $n \geqslant 3$. Then

- (Conn) $(X \cup Y, Y)$ is $(n-1)$-connected, and
- (Iso) $\pi_n(X \cup Y, Y) \cong \lambda_* \pi_n(X, X \cap Y)$, the $\pi_1(Y)$-module induced from the $\pi_1(X \cap Y)$-module $\pi_n(X, X \cap Y)$ by the morphism $\lambda : \pi_1(X \cap Y) \to \pi_1(Y)$ determined by inclusion.

The same applies for $n = 2$ with 'module' replaced by 'crossed module'.

Corollary: Classical Relative Hurewicz theorem
If the pair (X, A) is $(n-1)$-connected, $n \geqslant 2$, then $H_n(X, A)$ is isomorphic to $\pi_n(X, A)$ factored by the action of $\pi_1(A)$.

Proof: Take $Y = CA$.

NOTE: What happens in low dimensions affects what happens in high dimensions. What is going on?

Crossed modules: due to J.H.C. Whitehead

A central concept in nonabelian algebraic topology.

Morphism $\mu : M \to P$ of groups;
an action of the group P on the (right of) group M;
two rules for all $m, n \in M, p \in P$:

CM1) $\mu(m^p) = p^{-1}(\mu m)p$;

CM2) $m^{-1}nm = n^{\mu m}$.

Crossed modules should be thought of as
2-dimensional groups.
Classifying space $B(M \to P)$ with a homotopy fibration

$$BM \to BP \to B(M \to P),$$

so giving $\pi_i B(M \to P)$.

Crossed modules determine homotopy 2-types. The second homotopy group, even with the action of the fundamental group, is but a pale shadow of the homotopy 2-type.

Typical examples: algebraic ones have analogues for other structures than groups.

GEOMETRIC EXAMPLES:

1) $\Pi_2(X,A) = (\pi_2(X,A) \to \pi_1(A))$
second relative homotopy groups;

2) $\pi_1(F) \to \pi_1(E)$ when $F \to E \to B$ is a fibration sequence (useful in algebraic K-theory);

ALGEBRAIC EXAMPLES:

3) $M \lhd P$, i.e. M is normal subgroup of P;

4) $0 : M \to P$, i.e. M is a P-module;

5) $\chi : M \to Aut(M)$, the inner automorphism map;

6) $C(R) \to F(X)$, the free crossed module determined by a presentation $\langle X|R\rangle$ of a group G;

Homotopy classification:

If X is a reduced CW-complex, and M is a crossed module, then

$$[X, B\mathsf{M}] \cong [\Pi_2(X, X^1), \mathsf{M}],$$

a bijection of homotopy classes.

The next construction shows how new things occur for ‘2-dimensional groups’, corresponding to the geometry.

7) $\partial : \iota_*(M) \to Q$ induced from a crossed module $\mu : M \to P$ by a morphism $\iota : P \to Q$ of groups.

By the excision theorem, this last example for the crossed module $1 : P \to P$, gives the

homotopy 2-type of $Z = CB(\iota)$

when $\iota : P \to Q$ is a morphism of groups, so that $B(\iota) : BP \to BQ$. Not accessible any other way! To get information on the abelian π_2 we may need to go through a nonabelian route.

Projects: Apply crossed modules in low dimensional topology, TQFT (Yetter, Porter), HQFT (Turaev and Porter), differential topology (Mackaay, Brown/Icen), ...

Need combinatorial and computational crossed module theory. The computational side is largely due to Chris Wensley, Bangor, using GAP.

Computer calculations for $Q = S_4$ and various subgroups P of Q, with the crossed module $1 : P \to P$.

P	$\iota_* P$	π_1	π_2
$\langle (1,2) \rangle$	$GL(2,3)$	1	C_2
S_3	$GL(2,3)$	1	C_2
$\langle (1,2),(3,4) \rangle$	$S_4 C_2$	1	C_2
D_8	$S_4 C_2$	1	C_2
C_4	96.67	1	C_4
C_3	$C_3\ SL(2,3)$	C_2	C_6
$\langle (1,2)(3,4) \rangle$	128.2322	S_3	$C_4 C_2^3$

The computer has of course full information on the morphisms $\partial : \iota_*(P) \to S_4$ in terms of generators of the groups in the table. The third column gives $\pi_1 =$ Coker $\partial = \pi_1(Z)$. The fourth column gives $\pi_2 =$ Ker $\partial = \pi_2(Z)$. The numbers 96.67 and 128.2322 refer to particular groups of orders 96 and 128 respectively in the GAP4 table of groups.

NOTES:

1) Crossed complexes do not capture all homotopy types.

2) More powerfully, one can work similarly with n-cubes of spaces and cat^n-groups (Loday, Loday-RB). These structures do capture all pointed $(n+1)$-types. Lots unexplored!!

3) The van Kampen Theorems are limited in application. We can calculate the 3-type of SBG for G a group but not the 4-type.

4) General problem: follow Grothendieck's programme of applications in algebraic geometry and algebraic number theory!!!

5) Apply to differential geometry!!!

Proof of the excision theorem, particularly in the nonabelian case?

Back to the roots of algebraic topology, in homology and the fundamental group. Exposition of basic homotopy theory is tricky.

Topologists of the early 20th century knew:

- The nonabelian fundamental group $\pi_1(X, a)$ was important in analysis, geometry, and topology.
- Homology groups $H_n(X)$ existed in all dimensions, and were Abelian.
- X connected implies $H_1(X) \cong \pi_1(X, a)^{ab}$, the fundamental group made abelian.

So they dreamed of a higher dimensional generalisation of the fundamental group.

1932: Čech defined higher homotopy groups $\pi_n(X, a)$, $n \geqslant 2$, for the ICM at Zürich.

Alexandroff and Hopf proved these groups were Abelian and persuaded Čech to withdraw his paper.

We now see this as 'group objects internal to the category of groups are just abelian groups'.

Problem: to recapture the higher dimensional information.

I wrote a book in 1968 which showed to me that:

1-dimensional homotopy theory is better expressed in terms of groupoids rather than groups.

Groupoids include groups, group actions, equivalence relations.

The category of groupoids is more interesting than the category of groups!

Obvious question: How to use groupoids in higher dimensional homotopy theory?

CLUE: Groups internal to groupoids are equivalent to crossed modules (so are not just abelian groups)!

So we are looking at the extension of the usual theories:

$$(\text{groups}) \subseteq (\text{groupoids}) \subseteq (\text{multiple groupoids})$$

How useful will this be maths and science of the 21st century? Fun to investigate!

AMAZING FACT:
n-fold groupoids model homotopy n-types!

Calculate fundamental groupoid with a van Kampen theorem. Need a higher dimensional version!

Explain the 'linear' version of Brown/Higgins (1981) rather than the more powerful 'cubical' version of Brown/Loday (1987).

What to do next? Lots of problems:

1) Poincaré duality?

2) Applications to physics: crossed modules in TQFTs studied by Yetter and developed by Porter; in HQFTs are being studied by Tim Porter and Vladimir Turaev; applications to differential geometry (e.g. gerbes) studied by Brown/Glazebrook/Porter.

3) Develop applications of cat^n-groups and crossed n-cubes of groups.

4) Develop nonabelian homological algebra and apply it to algebraic geometry (Grothendieck's programme!).

See my paper 'Some problems ....' giving 35 problems or problem areas!