Crossed modules
and the homotopy 2-type of a free loop space

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Abstract

The question was asked by Niranjan Ramachandran, coming from Galois theory: how to describe the fundamental groupoid of LX, the free loop space of a space X? We show how this depends on the homotopy 2-type of X by assuming X to be the classifying space of a crossed module over a group, and then describe completely a crossed module over a groupoid determining the homotopy 2-type of LX; that is we describe crossed modules representing the 2-type of each component of LX. The method requires detailed information on the monoidal closed structure on the category of crossed complexes.[1]

1 Introduction

It is well known that for a connected CW-complex X with fundamental group G the set of components of the free loop space LX of X is bijective with the set of conjugacy classes of the group G, and that the fundamental groups of LX fit into a family of exact sequences derived from the fibration $LX \to X$ obtained by evaluation of a loop f at the point $0 \in S^1$.

Much interest in free loop space is in fact in the applications of their homology; work on this often uses information on the fundamental groups of LX, and the results here might yield homological information when combined with results of [Ell92, Ell95].

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In [Hans74], and in the case $\pi_2(X) = 0$, the fundamental group of LX based at a loop f is described as the centralizer of $[f]$ in $\pi_1(X, f(0))$:

$$\pi_1(LX, f) \cong C_{[f]}(\pi_1(X, f(0)).$$

This results from the fibration sequence $\Omega(X, f(0)) \to LX \to X$. We shall in Theorem 3.2 apply an analogous crossed module version of this fibration sequence to obtain an exact sequence giving information on the fundamental group in the case when $\pi_2(X) \neq 0$.

In fact our aim is stronger, namely to to describe completely the homotopy 2-type of LX, the free loop space on X, in terms of the 2-type of X, when X is a connected CW-complex. Weak homotopy 2-types are described by crossed modules (over groupoids), defined in [BH81b], generalising the pointed case in [MLW50].

A crossed module is the dimension 2 case of a *crossed complex*, the definition of which in the single vertex case goes back to Blakers in [Bla48], there called a 'group system', and in the many vertex case is in [BH81b]. The definition of the nerve of a crossed complex C in the one vertex case is also in [Bla48], and in the general case is in [Ash88, BH91]. An alternative description of K is that K_n consists of the crossed complex morphisms $\Pi\Delta^n_* \to \mathcal{M}$ where $\Pi\Delta^n_*$ is the fundamental crossed complex of the n-simplex, with its skeletal filtration, and $\mathcal{M}$ is also considered as a crossed complex trivial in dimensions > 2. This shows the analogy with the Dold-Kan theorem for chain complexes and simplicial abelian groups, [Dol58, BH03].

We thus define the *classifying space* $B\mathcal{M}$ *of* $\mathcal{M}$ to be the geometric realisation $|N^\Delta \mathcal{M}|$, a special case of the definition in [BH91]. It follows that an $a \in P(x)$ for some $x \in P_0$ determines a 1-simplex in $X = B\mathcal{M}$ which is a loop and so a map $a' : S^1 \to B\mathcal{M}$, i.e. $a' \in LX$.

The chief properties of $X = B\mathcal{M}$ are that $\pi_0(X) \cong \pi_0(P)$ and for each $x \in P_0$

$$\pi_i(X, x) \cong \begin{cases} \mathrm{Cok}(\delta : M(x) \to P(x)) & \text{if } i = 1, \\ \mathrm{Ker}(\delta : M(x) \to P(x)) & \text{if } i = 2, \\ 0 & \text{if } i > 2. \end{cases}$$

Further if Y is a CW-complex, then there is a crossed module $\mathcal{M}$ and a map $Y \to B\mathcal{M}$ inducing isomorphisms of π_0, π_1, π_2. For an exposition of some basic facts on crossed modules and crossed complexes in relation to homotopy theory, see for example [Bro99]. There are other versions of the classifying space, for example the cubical version given in [BHS11], and one for crossed module of groups using the equivalence of these with groupoid objects in groups, see for example [Lod82, BS08]. However the latter have not been shown to lead to the homotopy classification Theorem 1.1 below.

Our main result is Theorem 2.1, which is essentially an exercise in the use of the following classification theorem, [BH91, Theorem A], [BHS11, Theorem 11.4.19]:

Theorem 1.1 *Let Y be a CW-complex with its skeletal filtration Y_* and let C be a crossed complex, with its classifying space written BC. Then there is a natural weak homotopy equivalence*

$$B(\mathsf{CRS}(\Pi Y_*, C)) \to (BC)^Y.$$

In the statement of this theorem we use the internal hom $\mathsf{CRS}(-,-)$ in the category Crs of crossed complexes: this internal hom is described explicitly in [BH87], [BHS11, Section 9.3.1], in order to set up the exponential law

$$\mathsf{Crs}(A \otimes B, C) \cong \mathsf{Crs}(A, \mathsf{CRS}(B, C))$$

for crossed complexes A, B, C, i.e. to give a monoidal closed structure on the category Crs. Note that $\mathsf{CRS}(B,C)_0 = \mathsf{Crs}(B,C)$, $\mathsf{CRS}(B,C)_1$ gives the homotopies of morphisms, and $\mathsf{CRS}(B,C)_n$ for $n \geqslant 2$ gives the higher homotopies. The elements of $\mathsf{CRS}(B,C)$ are especially easy to write down in the case B is a free crossed complex, since the morphisms, homotopies and higher homotopies, are given by their values on the free basis.

2 Crossed modules

A *crossed module* $\mathcal{M}$ is a morphism $\delta : M \to P$ of groupoids which is the identity on objects such that M is just a disjoint union of groups $M(x), x \in P_0$, together with an action of P on M written $(m,p) \mapsto m^p$, $m \in M(x), p : x \to y$ with $m^p \in M(y)$ satisfying the usual rules for an action. We find it convenient to use (non-commutative) additive notation for composition so if $p : x \to y, q : y \to z$ then $p + q : x \to z$, and $(m+n)^p = m^p + n^p, (m^p)^q = m^{p+q}, m^0 = m$. Further we have the two crossed module rules for all $p \in P, m, n \in M$:

CM1) $\delta(m^p) = -p + \delta m + p$;

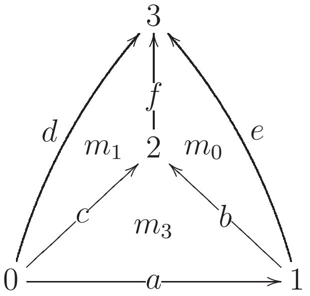
CM2) $-n + m + n = m^{\delta n}$;

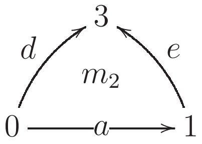
whenever defined. This is a *crossed module of groups* if P_0 is a singleton.

A crossed module $\mathcal{M}$ as above has a simplicial nerve $K = N^\Delta \mathcal{M}$ which in low dimensions is described as follows:

- $K_0 = P_0$;
- $K_1 = P$;
- K_2 consists of quadruples $\sigma = (m; c, a, b)$ where $m \in M, a, b, c \in P$ and $\delta m = -c + a + b$ is well defined;

- K_3 consists of quadruples $(\sigma_0, \sigma_1, \sigma_2, \sigma_3)$ where $\sigma_i \in K_2$ and the σ_i make up the faces of a 3-simplex, as shown in the following diagrams:





$$\delta m_2 = -d + a + e$$

providing we have the rules

$$\delta m_0 = -e + b + f, \quad \delta m_1 = -d + c + f,$$
$$\delta m_2 = -d + a + e, \quad \delta m_3 = -c + a + b,$$

together with the rule

$$(m_3)^f - m_0 - m_2 + m_1 = 0.$$

You may like to verify that these rules are consistent.

Our main result is:

Theorem 2.1 *Let $\mathcal{M}$ be the crossed module of groups $\delta : M \to P$ and let $X = B\mathcal{M}$ be the classifying space of $\mathcal{M}$. Then the components of LX, the free loop space on X, are determined by equivalence classes of elements $a \in P$ where a, b are equivalent if and only if there are elements $m \in M, p \in P$ such that*

$$b = p + a + \delta m - p.$$

Further the homotopy 2-type of a component of LX given by $a \in P$ is determined by the crossed module of groups $L\mathcal{M}[a] = (\delta_a : M \to P(a))$ where

(i) *$P(a)$ is the group of elements $(m, p) \in M \times P$ such that $\delta m = [a, p] = -a - p + a + p$, with composition $(n, q) + (m, p) = (m + n^p, q + p)$;*

(ii) *$\delta_a(m) = (-m^a + m, \delta m)$, for $m \in M$;*

(iii) *the action of $P(a)$ on M is given by $n^{(m,p)} = n^p$ for $n \in M, (m, p) \in P(a)$.*

In particular $\pi_1(LX,a)$ is isomorphic to $\operatorname{Cok}\delta_a$, and $\pi_2(LX,a)\cong\pi_2(X,*)^{\bar a}$, the elements of $\pi_2(X,*)$ fixed under the action of $\bar a$, the class of a in $G=\pi_1(X,*)$.

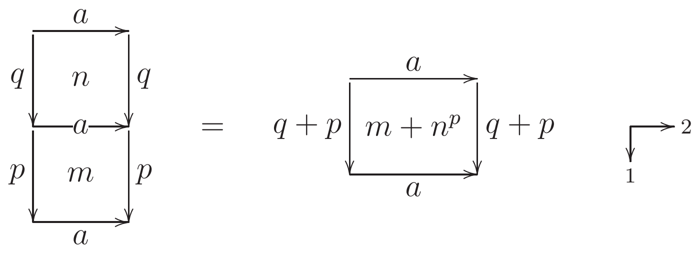
There is an exact sequence

$$\pi\xrightarrow{\phi}\pi\to\pi_1(LX,a)\to C_{\bar a}(\pi_1(X,*))\to 1,$$

in which $\pi=\pi_2(X,*)$, and ϕ is the morphism $m\mapsto -m^a+m$.

We give a detailed proof that $L\mathcal{M}[a]$ is a crossed module in Appendix 3.

Remark 2.2 The composition in (i) can be seen geometrically in the following diagram:



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Example 2.3 If $\delta=0:M\to P$, so that M is a P-module, then $P(a)$ is the set of (m,p) such that $[a,p]=0$, which is equivalent to $p\in C_a(P)$. So $P(a)=M\rtimes C_a(P)$, and $P=G$ the fundamental group, as $\delta=0$. But $\delta_a(m)=(-m^a+m,0)$. So

$$\pi_1(L\mathcal{M},a)=(M/[a,M])\rtimes C_a(P).$$

☐

Example 2.4 If $a\in Z(P)$, the center of P, then $[a,p]=0$ for all p. (For example, P might be abelian.) Hence $P(a)=\pi\rtimes P$. Then

$$\pi_1(L\mathcal{M},a)=(\pi\rtimes P)/\{(-m^a+m,\delta m)\mid m\in M\}.$$

☐

3 Proofs

We deduce Theorem 2.1 from the following Theorem.

Theorem 3.1 *Let $X=B\mathcal{M}$, where $\mathcal{M}$ is the crossed module of groups $\delta:M\to P$. Then the homotopy 2-type of LX, the free loop space of X, is described by the crossed module over groupoids $L\mathcal{M}$ where*

(i) $(L\mathcal{M})_0=P$;

(ii) $(L\mathcal{M})_1 = M \times P \times P$ *with source and target given by*

$$s(m,p,a) = p + a + \delta m - p, \quad t(m,p,a) = a$$

for $a, p \in P, m \in M$*;*

(iii) *the composition of such triples is given by*

$$(n,q,b) + (m,p,a) = (m + n^p, q + p, a)$$

which of course is defined under the condition that

$$b = p + a + \delta m - p$$

or, equivalently, $b^p = a + \delta m$*;*

(iv) *if* $a \in P$ *then* $(L\mathcal{M})_2(a)$ *consists of pairs* (m,a) *for all* $m \in M$*, with addition and boundary*

$$(m,a) + (n,a) = (m+n, a), \qquad \delta(m,a) = (-m^a + m, \delta m, a);$$

(v) *the action of* $(L\mathcal{M})_1$ *on* $(L\mathcal{M})_2$ *is given by:* $(n,b)^{(m,p,a)}$ *is defined if and only if* $b^p = a + \delta m$ *and then its value is* (n^p, a)*.*

Proof In Theorem 1.1 we set $Y = S^1$ with its standard cell structure $e^0 \cup e^1$, and can write $\Pi Y_* \cong \mathbb{K}(\mathbb{Z},1)$ where the latter is the crossed complex with a base point z_0 and a free generator z in dimension 1, and otherwise trivial. Thus morphisms of crossed complexes from $\mathbb{K}(\mathbb{Z},1)$, and homotopies and higher homotopies of such morphisms, are completely determined by their values on z_0 and on z.

A crossed module over a group or groupoid is also regarded as a crossed complex trivial in dimensions > 2.

All the formulae required to prove Theorem 3.1 follow from those for the internal hom CRS on the category Crs given in [BH87, Proposition 3.14] or [BHS11, §7.1.vii, §9.3].

We set $L\mathcal{M} = \mathsf{CRS}(\mathbb{K}(\mathbb{Z},1), \mathcal{M})$.

Since $\mathbb{K}(\mathbb{Z},1)$ is a free crossed complex with one generator z in dimension 1, the elements $a \in P$ are bijective with the morphisms $f : \mathbb{K}(\mathbb{Z},1) \to \mathcal{M}$, and we write this bijection as $a \mapsto \hat{a}$, where $a = \hat{a}(z)$. Also the homotopies and higher homotopies from $\mathbb{K}(\mathbb{Z},1) \to \mathcal{M}$ are determined by their values on z and on the element z_0 of $\mathbb{K}(\mathbb{Z},1)$ in dimension 0. Thus a 1-homotopy $(h, \hat{a}) : \hat{b} \simeq \hat{a}$ is such that h lifts dimension by 1, and is given by elements $p = h(z_0) \in P, m = h(z) \in M$ and so $(h, \hat{a})$ is given by a triple (p, m, a). The condition that this triple gives a homotopy $\hat{b} \simeq \hat{a}$ translates to

$$b = p + a + \delta m - p$$

or, equivalently, $a+\delta m = b^p$. It follows easily that $\hat{b}, \hat{a}$ belong to the same component of $L\mathcal{M}$ if and only if b, a give conjugate elements in the quotient group $\pi_1(\mathcal{M})$. (The use of such general homotopies was initiated in [Whi49].)

The composition of such homotopies $\hat{c} \simeq \hat{b} \simeq \hat{a}$ is given by:

$$(n, q, b) + (m, p, a) = (m + n^p, q + p, a)$$

which of course is defined if and only if

$$b^p = a + \delta m.$$

A 2-homotopy $(H, \hat{a})$ of $\hat{a}$ is such that H lifts dimension by 2 and so is given by an element $H(z_0) \in M$. There are rules giving the composition, actions, and boundaries of such 1- and 2-homotopies.

In particular the action of a 1-homotopy $(h, f^+) : f^- \simeq f^+$ on a 2-homotopy (H, f^-) gives a 2-homotopy (H^h, f^+) where $H^h(c) = H(c)^{h(tc)}$. Here we take $c = z_0$ so that we obtain the action $(n, b)^{(m,p,a)} = n^p$.

All these formulae follow from those given in [BH87, Proposition 3.14] or [BHS11, §9.3].

A 2-homotopy $(H, \hat{a})$ is given by $a = \hat{a}(z)$ and $m = H(z_0) \in M$. We then have to work out $\delta_2(H)$. We find that

$$\begin{aligned}\delta_2(H)(x) &= \begin{cases} \delta H(z_0) & \text{if } x = z_0, \\ -H(sz)^{\hat{a}(z)} + H(tz) + \delta H(z) & \text{if } x = z, \end{cases} \\ &= \begin{cases} \delta m & \text{if } x = z_0, \\ -m^a + m & \text{if } x = z. \end{cases}\end{aligned}$$

This completes the proof of Theorem 3.1. □

The proof of Theorem 2.1, apart from the exact sequence, now follows by restricting the crossed module of groupoids given in Theorem 3.1 to $L\mathcal{M}(a)$, the crossed module of groups over the object $a \in (L\mathcal{M})_1 = P$. Then we have an isomorphism $\theta : L\mathcal{M}(a) \to L\mathcal{M}[a]$ given by $\theta_0(a) = *$, $\theta_1(m, p, a) = (m, p)$, $\theta_2(m, a) = m$.

For the next result we need the notion of fibration of crossed modules of groupoids which is a special case of fibrations of crossed complexes as defined in [How79] and applied in [Bro08].

Theorem 3.2 *In the situation of Theorem 3.1, there is a fibration $L\mathcal{M} \to \mathcal{M}$ of crossed modules of groupoids. Hence if*

$$\pi = \pi_2(X) \cong \operatorname{Ker}\delta, \quad G = \pi_1(X) \cong \operatorname{Cok}\delta$$

then for each $a \in P$ there is an exact sequence

$$0 \to \pi^\alpha \to \pi \xrightarrow{\phi} \pi \to \pi_1(LX, a') \to C_{\bar{a}}(G) \to 1 \tag{1}$$

where π^α denotes the group of elements of π fixed under α, ϕ is the morphism $m \mapsto -m^\alpha - m$, and $C_\alpha(G)$ denotes the centraliser of the element $\alpha \in G$.

Proof We define the fibration $\psi : L\mathcal{M} \to \mathcal{M}$ by the inclusion $i : \{z_0\} \to \mathbb{K}(\mathbb{Z}, 1)$ and the identification $\mathsf{CRS}(\{z_0\}, \mathcal{M}) \cong \mathcal{M}$, where here $\{z_0\}$ denotes also the trivial crossed complex on the point z_0. Then ψ is a fibration since i is a cofibration, see [BG89]. The exact description of ψ in terms given earlier is that

$$\begin{aligned} \psi_0(a) &= *, \quad && a \in P, \\ \psi_1(m, p, a) &= p, \quad && (m, p, a) \in M \times P \times P, \\ \psi_2(n, a) &= n, \quad && (n, a) \in M \times P. \end{aligned}$$

To say that ψ is a fibration of crossed modules over groupoids is to say that: (i) it is a morphism; (ii) (ψ_1, ψ_0) is a fibration of groupoids, [Bro70, And78]; and (iii) ψ_2 is piecewise surjective.

We use the notion of exact sequence derived from a fibration of crossed complexes and in particular of crossed modules (of groupoids), [How79, Bro08, BHS11].

Let $\mathcal{F}$ denote the fibre of ψ. Then

$$\mathcal{F}_0 = P, \quad \mathcal{F}_1 = \{0\} \times M \times P, \quad \mathcal{F}_2 = \{0\} \times P.$$

The exact sequence of the fibration for a given base point $a \in \mathcal{F}_0 = P$ is

$$\begin{aligned} &0 \to \pi_2(\mathcal{F}, a) \to \pi_2(L\mathcal{M}, a) \to \pi_2(\mathcal{M}, *) \xrightarrow{\partial_2} \\ &\qquad \to \pi_1(\mathcal{F}, a) \to \pi_1(L\mathcal{M}, a) \to \pi_1(\mathcal{M}, *) \xrightarrow{\partial_1} \pi_0(\mathcal{F}) \to \pi_0(L\mathcal{M}) \to *. \end{aligned}$$

Note that $\mathcal{F}_1(a)$ is the set of $(m, 0, a) : a \to a$ and so $\delta m = 0$. Also $\mathcal{F}_2(a) = \{(0, a)\}$, and so

$$\pi_1(\mathcal{F}, a) \cong \pi = \pi_2(\mathcal{M}) = \operatorname{Ker} \delta.$$

Further, $\pi_2(L\mathcal{M}, a)$ is the set of (m, a) such that $\delta(m, a) = (-m^a + m, \delta m, a) = (0, 0, a)$. Hence $\pi_2(L\mathcal{M}, a) \cong \pi^a$, the set of elements of π fixed under the action of a.

We also have to evaluate the boundary morphism $\partial : \pi_2(\mathcal{M}) \to \pi_1(\mathcal{F}, a)$. Let $m \in M$ satisfy $\delta m = 0$. Then m lifts uniquely to an element $(m, a) \in L\mathcal{M}_2(a)$, and ∂m is given by $\delta(m, a) = (-m^a + m, 0, a)$. Thus under the above identifications, the boundary map ∂_2 becomes the morphism $\pi \to \pi, m \mapsto -m^a + m$.

Now we consider the boundary $\partial_1 : \pi_1(\mathcal{M}, a) \to \pi_0\mathcal{F}$, where the latter is given the base point $[a]$. An element $p \in P$ lifts to an element $(0, p, a)$ with target a, and source $p + a - p$, which is a loop if and only if $p \in C_a(P)$. By exactness, the image of ∂_1 is the claimed centraliser. This leads to the exact sequence of Theorem 2.1. □

Remark 3.3 These results and methods should be related to the description in [Bro87, §6] of the homotopy type of the function space $(BG)^Y$ where G is an abstract group and Y is a CW-complex, and which gives a result of Gottlieb in [Got69]. □

Remark 3.4 Here is a methodological point. The category Crs of crossed complexes is equivalent to that of ∞-groupoids, as in [BH81a], where these ∞-groupoids are now commonly called 'strict globular ω-groupoids'. However the internal hom in the latter category is bound to be more complicated than that for crossed complexes, because the cell structure of the standard n-globe, $n > 1$,

$$E^n = e^0_\pm \cup e^1_\pm \cup \cdots \cup e^{n-1}_\pm \cup e^n$$

is more complicated than that for the standard cell for which

$$E^n = e^0 \cup e^{n-1} \cup e^n, n > 1.$$

Also we obtain a precise answer using filtered spaces and strict structures, whereas the current fashion is to go for weak structures as yielding more homotopy n-types for $n > 2$. In fact many results on crossed complexes are obtained using cubical methods. □

Appendix: Verification of crossed module rules

We now verify the crossed module rules for the structure

$$L\mathcal{M}[a] = (M \xrightarrow{\delta_a} P(a))$$

defined in Theorem 2.1 from a crossed module of groups $\mathcal{M} = (M \xrightarrow{\delta} P)$ and $a \in P$ as follows:

$$\begin{aligned} P(a) &= \{(m, p) \in M \times P \mid \delta m = -[a, p] = -a - p + a + p\}; \\ \delta_a m &= (-m^a + m, \delta m); \\ (n, q) + (m, p) &= (m + n^p, q + p); \\ n^{(m,p)} &= n^p. \end{aligned}$$

Proposition 3.5 *If $\delta : M \to P$ is a crossed module of groups, and $a \in P$, then $L\mathcal{M}[a]$ as defined above is also a crossed module of groups.*

Proof It is easy to check that $\delta(-m^a+m)=[a,\delta m]$, so that $\delta_a(m)\in P(a)$.

We next show that δ_a is a morphism:

$$\begin{aligned}\delta_a(n)+\delta_a(m) &= (-n^a+n,\delta n)+(-m^a+m,\delta m)\\ &= (-m^a+m+(-n^a+n)^{\delta m},\delta n+\delta m)\\ &= (-m^a-n^a+n+m,\delta n+\delta m)\\ &= \delta_a(n+m).\end{aligned}$$

Now we verify the first crossed module rule. Let $(m,p)\in P(a), n\in M$:

$$\begin{aligned}-(m,p)+\delta_a n+(m,p) &= (-m^{-p},-p)+(-n^a+n,\delta n)+(m,p)\\ &= (-n^a+n+(-m^{-p})^{\delta n},-p+\delta n)+(m,p)\\ &= (-n^a-m^{-p}+n,-p+\delta n)+(m,p)\\ &= (m+(-n^a-m^{-p}+n)^p,-p+\delta n+p)\\ &= (m-n^{a+p}-m+n^p,\delta(n^p))\\ &= (-n^{a+p-\delta m}+n^p,\delta(n^p))\\ &= (-n^{p+a}+n^p,\delta(n^p)) && \text{since } \delta m=[a,p]\\ &= \delta_a(n^p).\end{aligned}$$

Now we verify the second crossed module rule:

$$\begin{aligned}m^{\delta_a n} &= m^{(-n^a+n,\,\delta n)}\\ &= m^{\delta n}\\ &= -n+m+n.\end{aligned}$$

□

In effect, this illustrates that verifying the crossed complex rules for the internal hom $\mathsf{CRS}(C,D)$ is possible but tedious, and that is it is easier to say it follows from the general construction in terms of ω-groupoids and the equivalence of categories, as in [BH87, BHS11]. On the other hand, this direct proof 'proves', in the old sense of 'tests', the general theory.

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