

# Covering morphisms of groupoids, derived modules and a 1-dimensional Relative Hurewicz Theorem\*

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## Abstract

We fill a lacuna in the literature by giving a version in dimension 1 of the Relative Hurewicz Theorem, and relate this to abelianisations of groupoids, covering spaces, covering morphisms of groupoids, and Crowell's notion of derived modules.

## Introduction

A classical result is that when the space  $X$  with base point  $a$  is path connected then the canonical morphism

$$\pi_1(X, a) \rightarrow H_1(X)$$

is just abelianisation of the fundamental group  $\pi_1(X, a)$  of  $X$  at  $a$ .

However there is a notion of the fundamental groupoid  $\pi_1(X, A)$  on a set  $A$  of base points, introduced in [Bro67] to allow for an associated Seifert–van Kampen Theorem for non connected spaces, and this concept is clearly also necessitated in discussing spaces with a group of operators, as developed in [HT82] and [Bro06, Chapter 11].

Our main aim in this paper is to give circumstances which enable us to identify the canonical morphism

$$\pi_1(X, A) \rightarrow H_1(X, A)$$

as an abelianisation of the groupoid  $\pi_1(X, A)$ . This identification is clearly a 1-dimensional form of the Relative Hurewicz Theorem. We then use covering morphisms of groupoids to relate this result to Crowell's notion of derived module, [Cro71], and show its relevance to covering spaces.

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The usual Relative Hurewicz Theorem (RHT) states that if  $n \geq 2$ , and  $(X, A, x)$  is an  $(n - 1)$ -connected pair of connected spaces, then the natural Hurewicz morphism in dimension  $n$

$$\pi_n(X, A, x) \rightarrow H_n(X, A)$$

is given by factoring the action of  $\pi_1(A, x)$ . This result is basic in algebraic topology. It has been rephrased in [BH81, Section 8], see also [BS09], as the statement that under the same conditions the natural morphism

$$\pi_n(X, A, x) \rightarrow \pi_n(X \cup CA, CA, x)$$

has the same properties: the proofs of these results are given in [BH81, BHS11] independently of homology theory, as a special case of a Higher Homotopy Seifert-van Kampen Theorem, involving pushouts, or more general colimits. This more general theorem even has a version involving a set  $X_0$  of base points, for which one has to use the notion of induced module or crossed module over groupoids for a precise statement of a theorem analogous to the RHT, see for example [BS09].

## 1 Derived modules

In this section we recall the work of Crowell in [Cro71]. Let  $\phi : F \rightarrow G$  be a morphism of groups, and let  $M$  be a (right)  $G$ -module.

**Definition 1.1** A  $\phi$ -derivation  $d : F \rightarrow M$  is a function satisfying

$$d(uv) = d(u)^{\phi v} + d(v)$$

for all  $u, v \in F$ .

A  $\phi$ -derivation  $\partial : F \rightarrow D$  is said to be a *universal  $\phi$ -derivation* if it is a  $\phi$ -derivation and for any other  $\phi$ -derivation  $d : F \rightarrow M$  there is a unique  $G$ -morphism  $d' : D \rightarrow M$  such that  $d'\partial = d$ .

In case  $\phi$  is the identity morphism then these terms are abbreviated by omitting the  $\phi$ . An example of a derivation is  $e : G \rightarrow IG$ ,  $g \mapsto g - 1$ , where  $IG$  is the augmentation ideal of  $G$ . □

**Example 1.2** An example of a universal derivation is  $e : G \rightarrow IG$  where  $IG$  is the augmentation ideal, which is generated as  $G$ -module by the elements  $g - 1, g \neq 1, g \in G$  and  $e$  is given by  $g \mapsto g - 1$ . □

Since a universal  $\phi$ -derivation  $\partial : F \rightarrow D$  determines  $D$  uniquely up to isomorphism, the  $G$ -module  $D$  is then called the *derived module* of the morphism  $\phi$ , and is written  $D_\phi$ . The derived module may be constructed directly and also as  $IF \otimes_{\mathbb{Z}F} \mathbb{Z}G$ .

One of the key uses of the derived module is in the following exact sequence of Crowell, [Cro71, Section 4].

**Proposition 1.3** *Let  $\phi : F \rightarrow G$  be an epimorphism of groups, with kernel  $N$ . Then there is an exact sequence of  $G$ -modules*

$$0 \rightarrow N^{\text{ab}} \rightarrow D_\phi \rightarrow IG \rightarrow 0. \quad (1)$$

**Remark 1.4** This construction of an exact sequence of modules from an exact sequence of groups is the key to relating the classification of extensions of groups to the homological algebra of modules, and is treated in [ML63]. The construction of the derived module is generalised to the groupoid case in [BH90], as is essential for the results there, and the above sequence is also generalised.  $\square$

## 2 Abelianisations of groupoids

We use at several points the well known abelianisation of a group. This construction gives a functor

$${}^{\text{ab}}: \text{Groups} \rightarrow \text{Ab}$$

which is left adjoint to the inclusion of categories

$$\text{Ab} \rightarrow \text{Groups}.$$

We define a groupoid  $G$  to be *abelian* if all its vertex groups are abelian groups. The *abelianisation of a groupoid*  $G$ , which we write  $G^{\text{ab}}$ , is obtained by quotienting  $G$  with the normal subgroupoid generated by the commutators from all vertex groups.

Nonetheless, we need another kind of abelianisation that associates to each groupoid not an abelian groupoid but an abelian group and a morphism  $v: G \rightarrow G^{\text{totab}}$  which is universal for morphisms to abelian groups. We call  $G^{\text{totab}}$  the *universal abelianisation* of the groupoid  $G$ .

Let  $\text{Ab}$ ,  $\text{Groups}$ ,  $\text{Gpd}$  denote respectively the categories of abelian groups, groups, and groupoids. Each of the inclusions

$$\text{Ab} \rightarrow \text{Groups} \rightarrow \text{Gpd} \quad (2)$$

has a left adjoint. That from groupoids to groups is called the *universal group*  $UG$  of a groupoid  $G$  and is described in detail in [Hig71] and [Bro06, Section 8.1]. In particular, the universal group of a groupoid  $G$  is the free product of the universal groups of the transitive components of  $G$ . Any transitive groupoid  $G$  may be written in a non canonical way as the free product  $G(a_0) * T$  of a vertex group  $G(a_0)$  and an indiscrete or tree groupoid  $T$ . Then

$$UG \cong G(a_0) * UT$$

and  $UT$  is the free group on the elements  $x: a_0 \rightarrow a$  in  $T$  for all  $a \in \text{Ob}(T)$ ,  $a \neq a_0$ .

It follows that the universal abelianisation is given by

$$G^{\text{totab}} \cong (UG)^{\text{ab}}, \quad (3)$$

and also that  $G^{\text{totab}}$  is isomorphic to the direct sum of the  $G_i^{\text{totab}}$  over all components  $G_i$  of  $G$ . So for a transitive groupoid  $G$  with  $a_0 \in \text{Ob } G$

$$G^{\text{totab}} \cong G(a_0)^{\text{ab}} \oplus F \quad (4)$$

where  $F$  is the free abelian group on the elements  $x: a_0 \rightarrow a$  in  $T$  for all  $a \in \text{Ob}(T), a \neq a_0$ , for  $T$  a wide tree subgroupoid of  $G$ .

If  $G$  is a totally disconnected groupoid on the set  $X$ , and  $\theta: G \rightarrow X$  is the unique morphism over  $X$  to the discrete groupoid on  $X$ , then the derived module of  $\theta$  is the above defined abelianisation  $G^{\text{ab}}$  of  $G$ .

### 3 Covering morphisms of groupoids

For the convenience of readers, and to fix the notation, we recall here the basic facts on covering morphisms of groupoids. The earliest definition of covering morphism of groupoids seems to be in [Smi51a, Smi51b], where such a morphism is called a ‘regular’ morphism. The ideas were developed independently in [Hig64]. Many basic facts which should be seen as a part of ‘combinatorial groupoid theory’ can be found in the books [Hig71, Bro06]. However we find it convenient to adopt different conventions from, say, [Bro06], focussing on costars rather than stars.

Let  $G$  be a groupoid. For each object  $a_0$  of  $G$  the *Costar* of  $a_0$  in  $G$ , denoted by  $\text{Cost}_G a_0$ , is the union of the sets  $G(a, a_0)$  for all objects  $a$  of  $G$ , i.e.  $\text{Cost}_G a_0 = \{g \in G \mid tg = a_0\}$ . A morphism  $p: \tilde{G} \rightarrow G$  of groupoids is a *covering morphism* if for each object  $\tilde{a}$  of  $\tilde{G}$  the restriction of  $p$

$$\text{Cost}_{\tilde{G}} \tilde{a} \rightarrow \text{Cost}_G p\tilde{a} \quad (5)$$

is bijective. In this case  $\tilde{G}$  is called a *covering groupoid* of  $G$ . More generally  $p$  is called a *fibration of groupoids*, [Bro70], if the restrictions of  $p$  to the Costars as in Equation (5) are surjective.

A basic result for covering groupoids is *unique path lifting*. That is, let  $p: \tilde{G} \rightarrow G$  be a covering morphism of groupoids, and let  $(g_1, g_2, \dots, g_n)$  be a sequence of composable elements of  $G$ . Let  $\tilde{a} \in \text{Ob}(\tilde{G})$  be such that  $p\tilde{a}$  is the target of  $g_n$ . Then there is a unique composable sequence  $(\tilde{g}_1, \tilde{g}_2, \dots, \tilde{g}_n)$  of elements of  $\tilde{G}$  such that  $\tilde{g}_n$  ends at  $\tilde{a}$  and  $p\tilde{g}_i = g_i, i = 1, \dots, n$ .

If  $G$  is a groupoid, the slice category  $\text{GpdCov}/G$  of coverings of  $G$  has as objects the covering morphisms  $p: H \rightarrow G$  and has as morphisms the commutative diagrams of

morphisms of groupoids, where  $p$  and  $q$  are covering morphisms,

$$\begin{array}{ccc} H & \xrightarrow{f} & K \\ & \searrow p & \swarrow q \\ & & K \end{array}$$

By a standard result on compositions and covering morphisms ([Bro06, 10.2.3]),  $f$  also is a covering morphism. It is convenient to write such a diagram as a triple  $(f, p, q)$ . The composition in  $\mathbf{GpdCov}/G$  is then given as usual by

$$(g, q, r)(f, p, q) = (gf, p, r).$$

It is a standard result (see for example [Hig71, Bro70]) that the category  $\mathbf{GpdCov}/G$  is equivalent to the category of operations of the groupoid  $G$  on sets. We give the definitions and notations which we will use for this equivalence.

Recall we are writing composition of  $g: p \rightarrow q$  and  $h: q \rightarrow r$  in a groupoid as  $gh: p \rightarrow r$ . This is the opposite of the notation for functions in the category  $\mathbf{Set}$ ; the composite of a function  $f: X \rightarrow Y$  and  $g: Y \rightarrow Z$  is  $gf: X \rightarrow Z$  with value  $(gf)(x) = g(f(x))$ . It is possible to resolve this confusion by writing functions on the right of their argument as  $(x)f$ : This ‘algebraist’s’ convention is followed successfully in [Hig71], and contrasts with the usual ‘analyst’s’ convention. Because of this ‘opposite’ nature of our conventions we have to make the following definition.

**Definition 3.1** A *left operation of a groupoid  $G$  on sets* is a functor  $X: G^{\text{op}} \rightarrow \mathbf{Set}$ . If  $p \in \text{Ob}(G)$ ,  $g: p \rightarrow q$  in  $G$ , and  $x \in X(p)$ , then  $X(g)(x) \in X(q)$  may also be written  $g \cdot x$ .

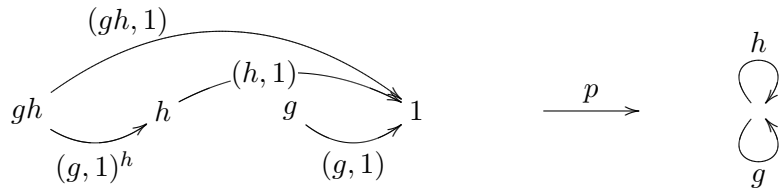
Thus if  $X: G^{\text{op}} \rightarrow \mathbf{Set}$  is a functor, then the *action groupoid*  $\tilde{G} = G \times X$  has object set the disjoint union of the sets  $X(p)$  for  $p \in \text{Ob}(G)$  and morphisms  $x \rightarrow y$  the pairs  $(g, x)$  such that  $x \in X(p)$  and  $y = X(g)x$ ; in operator notation:  $(g, x): gx \rightarrow x$ . The composition is  $(g', gx)(g, x) = (g'g, x)$ . The projection morphism  $G \times X \rightarrow G$ ,  $(g, x) \mapsto g$ , is a covering morphism.

This ‘semidirect product’ or ‘Grothendieck construction’ is fundamental for constructing covering morphisms to the groupoid  $G$ . This so called ‘Grothendieck construction’ has also been developed by C. Ehresmann in [Ehr57], in which he defines both an action of a category and the associated ‘category of hypermorphisms’, and also what in the case of local groupoids he calls the complete enlargement of a species of structures. For example, if  $a_0$  is an object of the transitive groupoid  $G$ , and  $A$  is a subgroup of the object group  $G(a_0)$  then the groupoid  $G$  operates on the family of cosets  $\{gA \mid g \in \text{Cost}_G a_0\}$ , by  $g'.(gA) = g'gA$  whenever  $g'g$  is defined, and the associated covering morphism  $\tilde{G} \rightarrow G$  defines the covering groupoid  $\tilde{G}$  of the groupoid  $G$  determined by the subgroup  $A$ . When  $A$  is trivial this gives the *universal cover* at  $a_0$  of the groupoid  $G$ .

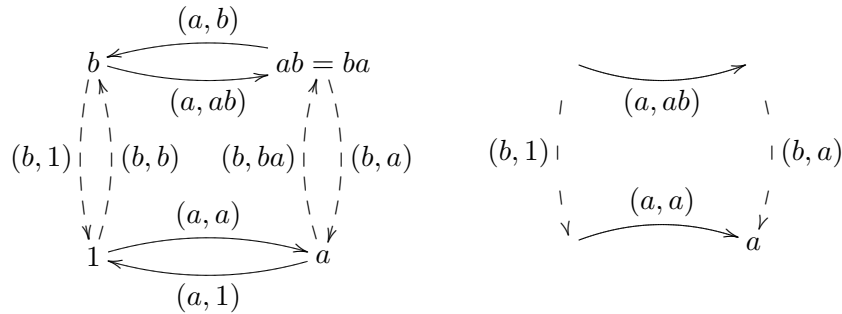
In particular, this gives the universal covering groupoid of a group  $G$ , whose objects are the elements of  $G$  and morphisms are pairs  $(g, h): gh \rightarrow h$ ,  $g, h \in G$ , and composition is

$$(g, hk)(h, k) = (gh, k)$$

for  $g, h, k \in G$ . Then  $G$  operates on the right of the universal cover by  $(g, h)^k = (g, hk)$ , for  $g, h, k \in G$ . This operation preserves the map  $p$  and is called a *covering transformation*. The relevance of these ideas to derivations is as follows. Let  $d: G \rightarrow \tilde{G}$  be defined by  $g \mapsto (g, 1)$ . Then  $d$  satisfies  $d(gh) = d(g)^h d(h)$  as illustrated by the following diagram:



**Example 3.2** Here is a simple example: the universal covering groupoid  $\tilde{K}$  of the Klein 4-group  $K = C_2 \times C_2$  with elements say  $1, a, b, ab$ . This group is generated by  $a, b$  with the relations  $a^2, b^2, aba^{-1}b^{-1}$ , which we write respectively  $r, s, t$ . Then  $\tilde{K}$  has the elements of  $K$  as vertices and a morphism  $(g, x): gx \rightarrow x$  for each  $g, x \in K$ . The covering morphism  $p: \tilde{K} \rightarrow K$  is  $(g, x) \mapsto g$ . In terms of the generators  $a, b$  we obtain a diagram of  $\tilde{K}$  as the left hand diagram in the following picture:



Note that for example  $(a, ab): b \rightarrow ab$  because  $a^2 = 1$ . The right hand diagram illustrates a lift of the loop  $b^{-1}a^{-1}ba$  in  $K$  to a loop starting and ending at  $a$  in the diagram of  $\tilde{K}$ . You should draw the similar loops starting in turn at  $1, b, ab$ . It is also possible to view these four loops as forming boundaries of four ‘lifts’ of the relation  $t$ .  $\square$

For a space  $X$  which is locally path-connected and semi-locally 1-connected there is a standard analysis of covering spaces of  $X$  in terms of operations of the fundamental groupoid  $\pi_1 X$  on sets, and this is the formalism usually followed in texts on algebraic topology; however there is another formulation of this result, which goes back in the

simplicial set case to [GZ67], and states that under the above local conditions on  $X$  the fundamental groupoid functor

$$\pi_1 : \mathbf{TopCov}/X \rightarrow \mathbf{GpdCov}/\pi_1 X \quad (6)$$

is an equivalence of categories. The construction of the inverse equivalence in Equation (6) is of course closely related to the usual method of constructing covering spaces, but seems to fall more naturally as part of this theorem, since it relates morphisms in two different contexts, namely spaces and groupoids, and the proofs are base point free. We indicate how the inverse equivalence may be constructed.

Let then  $q : \tilde{G} \rightarrow \pi_1 X$  be a covering morphism of groupoids. Let  $\tilde{X} = \text{Ob}(\tilde{G})$  and let  $p = \text{Ob}(q) : \tilde{X} \rightarrow X$ . One uses the properties of covering morphisms of groupoids and the assumed local conditions on  $X$  to lift sufficiently small neighbourhoods of points of  $X$  to neighbourhoods of points of  $\tilde{X}$  to obtain a base for a topology on  $\tilde{X}$  with the required properties. Full details of this construction are in [Bro06, Sections 10.5,10.6], and indeed were given in the 1968 edition of that book.

The above equivalence is used in [BM94] to discuss covering groups of non-connected topological groups.

## 4 Covering morphisms and derived modules

In this section we link the notions of covering morphisms with those of derived modules, in effect clarifying and giving algebraic versions of results in [Cro71, Section 5] and [Whi49, Section 11].

Let  $\phi : F \rightarrow G$  be a morphism of groups with kernel  $N$ . We form the universal covering groupoid  $p : \tilde{G} \rightarrow G$  and the pullback

$$\begin{array}{ccc} \hat{F} & \xrightarrow{\bar{\phi}} & \tilde{G} \\ q \downarrow & & \downarrow p \\ F & \xrightarrow{\phi} & G \end{array} \quad (7)$$

Note that a morphism in  $\hat{F}$  is a pair  $(u, (\phi u, g)) : (\phi u)g \rightarrow g$ ,  $u \in F, g \in G$ . Since  $u$  determines  $\phi u$ , we can write a morphism of  $\hat{F}$  as  $(u, g) : (\phi u)g \rightarrow g$ . One utility of this diagram is that if  $F$  is the free group on a set  $S$  of generators, and  $\phi$  is an epimorphism, then the graph  $q^{-1}(S)$  can be seen as the Cayley graph of  $G$  for this set of generators, and  $\hat{F}$  is the free groupoid on this graph. The free groupoid on a graph was introduced in [Hig64] and a recent use is in [CP01].

The group  $G$  operates on the right of  $\tilde{G}$  by  $(h, g)^k = (h, gk)$ ,  $k \in G$  and so on the right of  $\hat{F}$  by  $(u, g)^k = (u, gk)$ ,  $k \in G$ .

It is easy to prove, and alternatively follows from the Mayer-Vietoris sequence of the pullback, [Bro06, 10.7.6] and [BHK83], that  $\widehat{F}$  is connected if and only if  $\phi$  is surjective, and that  $q$  maps the vertex group  $\widehat{F}(1)$  bijectively to  $N$ .

**Theorem 4.1** *If  $\phi : F \rightarrow G$  is an epimorphism of groups with kernel  $N$  then there is a natural isomorphism of exact sequences of  $G$ -modules*

$$\begin{array}{ccccccc}
0 & \longrightarrow & N^{\text{ab}} & \longrightarrow & D_\phi & \longrightarrow & IG \longrightarrow 0 \\
& & \downarrow & & \downarrow \eta & & \downarrow \\
0 & \longrightarrow & N^{\text{ab}} & \longrightarrow & \widehat{F}^{\text{totab}} & \longrightarrow & \widetilde{G}^{\text{totab}} \longrightarrow 0.
\end{array} \tag{8}$$

**Proof** The top line of the diagram is the standard exact sequence of modules derived from the exact sequence of groups  $1 \rightarrow N \xrightarrow{\phi} F \rightarrow G \rightarrow 1$  as shown in [Cro71, Section 5] and which corresponds to a classical derivation of such a module sequence, [ML63, Section IV.6].

Here  $D_\phi$  is defined by a universal  $\phi$ -derivation  $d : F \rightarrow D_\phi$ . Thus to construct  $\eta$  we need a  $\phi$ -derivation  $e : F \rightarrow \widehat{F}^{\text{totab}}$ . This is the composition  $F \xrightarrow{f} \widehat{F} \xrightarrow{a} \widehat{F}^{\text{totab}}$ , where  $f$  is the function  $u \mapsto (u, 1)$ . Then

$$f(uv) = (uv, 1) = (uv, v)(v, 1) = (u, 1)^{\phi v}(v, 1)$$

from which it follows that  $e$  is a  $\phi$ -derivation.

Now the vertex group  $\widehat{F}(1)$  is isomorphic to  $N$  under  $p$  so we identify these. Also  $\text{Ob}(\widetilde{G}) = G$  and so, since  $\widehat{F}$  is connected,

$$\widehat{F}^{\text{totab}} \cong N^{\text{ab}} \oplus (G \times G)^{\text{totab}}.$$

However  $(G \times G)^{\text{totab}}$  is the free abelian group on elements  $g \in G, g \neq 1$ , by Equation (4). However in keeping with the module action of  $G$  this can also be regarded as the free abelian group on elements  $g - 1, g \neq 1$  where

$$hg - 1 = (h - 1)^g + (g - 1).$$

This is another description of the augmentation module  $IG$ .

Thus we have the map of exact sequences as given in the Theorem, and so by the 5-lemma,  $\eta$  is an isomorphism.  $\square$

## 5 Homology and homotopy

The homology groups of a cubical set  $K$  are defined as follows. First we form the chain complex  $C'(K)$  where  $C'_n(K)$  is the free abelian group on  $K_n$ , and with boundary

$$\partial k = \sum_{i=1}^n (-1)^i (\partial_i^- k - \partial_i^+ k). \quad (9)$$

It is easily verified that this gives a chain complex, i.e.  $\partial\partial = 0$ . However if  $K$  is a point, i.e.  $K_n$  is a singleton for all  $n$ , then the homology groups of  $C'(K)$  are  $\mathbb{Z}$  in even dimensions, whereas we want the homology of a point to be zero in dimensions  $> 0$ . We therefore normalise, i.e. factor  $C'(K)$  by the subchain complex generated by the degenerate cubes. This gives the chain complex  $C_*(K)$  of  $K$ , and the homology groups of this chain complex are defined to be the homology groups of  $K$ . In particular the (cubical) singular homology groups of the space  $X$  are defined to be the homology groups of the cubical singular complex  $S^\square X$ .

A full exposition of this cubical homology theory is in [HW60, Mas80]. It is proved in [EML53] using acyclic models that the cubical singular homology groups are isomorphic to the simplicial singular homology groups. Recent works using cubical methods are [GNAPGP88, BJT10, Isa09].

Let  $X_*$  be a filtered space: that is,  $X_*$  is given by a space  $X$  and a sequence of subspaces

$$X_*: = X_0 \subseteq X_1 \subseteq X_2 \subseteq \cdots \subseteq X_n \subseteq \cdots \subseteq X$$

which we call a *filtration* of  $X$ . A map of filtered spaces  $f: X_* \rightarrow Y_*$  is a map  $f: X \rightarrow Y$  preserving the filtration.

**Example 5.1** 1. A standard example of a filtered space is a *CW*-complex with its skeletal filtration. So among these we have the  $n$ -cubes  $I_*^n$  with their usual cell structure and skeletal filtration.

2. Let  $A$  be a subspace of the topological space  $X$ , and let  $n \geq 0$ . Then we have a filtration of  $X$  which we could write  $X_A^n$  defined by  $(X_A^n)_i = A$  for  $i \leq n$  and  $(X_A^n)_i = X$  for  $i > n$ . These filtrations occur in proofs of the Hurewicz Theorem.
3. Another example is the free monoid  $JX$  on a topological space  $X$ , with  $JX$  filtered by the length of the words. And finally, a filtration on a manifold is determined by a Morse function on the manifold.  $\square$

The notion of map of filtered spaces allows us to define the *filtered singular complex*  $RX_*$  which in dimension  $n$  is the set of filtered maps  $I_*^n \rightarrow X_*$ . This has the structure of cubical set, and other structure such as connections and compositions, which are discussed in [BH80]. Also  $RX_*$  is a fibrant cubical set, or in other terms, is a Kan complex, as is

easily proved: the use of this structure as a foundation for algebraic topology is described in [BHS11].

The cubical chain complex  $C_*(X_*)$  is defined to be the normalised chain complex of the cubical set  $RX_*$ .

The following retraction Proposition, which is taken from [BH81, Section 9], is one step towards the Hurewicz theorem. This retraction proposition should be compared with a special case discussed in [Mas80, Section III.7]. The history of classical papers on singular homology and the Hurewicz Theorem shows the use of deformation theorems of the type of this Proposition, as for example in Blakers [Bla48]. However the use of cubical methods as here, rather than of simplicial methods and chain complexes, does seem to simplify the proof, partly since cubical methods are easier for constructing homotopies.

In the proof, a useful lemma is that if  $(Y, Z)$  is a cofibred pair, and  $f: (Y, Z) \rightarrow (X, A)$  is a map of pairs which is deformable (as a map of pairs) into  $A$ , then  $f$  is deformable into  $A$  rel  $Z$  ([Bro06, 7.4.4]).

**Proposition 5.2** *Let  $X_*$  be a filtered space such that the following conditions  $\psi(X_*, m)$  hold for all  $m \geq 0$ :*

$\psi(X_*, 0)$  : *The map  $\pi_0 X_0 \rightarrow \pi_0 X$  induced by inclusion is surjective;*

$\psi(X_*, 1)$  : *Any path in  $X$  joining points of  $X_0$  is deformable in  $X$  rel end points to a path in  $X_1$ ;*

$\psi(X_*, m)(m \geq 2)$  : *For all  $\nu \in X_0$ , the map*

$$\pi_m(X_m, X_{m-1}, \nu) \rightarrow \pi_m(X, X_{m-1}, \nu)$$

*induced by inclusion is surjective.*

*Then the inclusion  $i: RX_* \rightarrow KX = S^{\square}X$  is a homotopy equivalence of cubical sets.*

**Proof** There exist maps  $h_m: K_m X \rightarrow K_{m+1} X$ ,  $r_m: K_m X \rightarrow K_m X$  for  $m \geq 0$  such that

- (i)  $\partial_{m+1}^- h_m = 1, \partial_{m+1}^+ h_m = r_m$ ;
- (ii)  $r_m(KX) \subseteq R_m X_*$  and  $h_m | R_m X_* = \varepsilon_{m+1}$ ;
- (iii)  $\partial_i^\tau h_m = h_{m-1} \partial_i^\tau$  for  $1 \leq i \leq m$  and  $\tau = -, +$ ;
- (iv)  $h_m \varepsilon_j = \varepsilon_j h_{m-1}$  for  $1 \leq j \leq m$ .

Such  $r_m, h_m$  are easily constructed by induction, starting with  $h_{-1} = \emptyset$ , and using  $\psi(X_*, m)$  to define  $h_m \alpha$  for elements  $\alpha$  of  $K_m X$  which are not degenerate and do not

lie in  $R_m X_*$ . Here is a picture for  $h_1$ :

$$\begin{array}{ccccc}
 & & h_0 \partial_1^- k & & \\
 & & \text{---} & & \\
 r \partial_1^- k & \bullet & \text{---} & \bullet & \partial_1^- k \\
 & & | & & | \\
 & & r k & & k \\
 & & | & & | \\
 & & h_1 k & & \\
 & & \text{---} & & \\
 r \partial_1^+ k & \bullet & \text{---} & \bullet & \partial_1^+ k \\
 & & h_0 \partial_1^+ k & & 
 \end{array}$$

These maps define a retraction  $r: KX \rightarrow RX_*$  and a homotopy  $h \simeq ir \text{ rel } RX_*$ .  $\square$

**Corollary 5.3** *If  $X_*$  is the skeletal filtration of a CW-complex, then the inclusion  $RX_* \rightarrow S^\square X$  is a homotopy equivalence of fibrant cubical sets.*  $\square$

**Corollary 5.4** *If the conditions  $\psi(X_*, m)$  of the proposition hold for all  $m \geq 0$ , then the inclusion  $i: RX_* \rightarrow S^\square X$  induces a homotopy equivalence of chain complexes and hence an isomorphism of all homology and homotopy groups.*

**Proof** The result on homotopy is standard, and that on homology follows from the development in [Mas80].  $\square$

## 5.1 Relative Hurewicz Theorem: dimension 1

In this subsection we identify the total abelianisation of the groupoid  $\pi_1(X, A)$ , see Section 2, in certain cases in terms of homology.

**Definition 5.5** We now coin a term: for a subspace  $A$  of  $X$ , let  $C_*(X \text{rel}_0 A)$  denote the sub chain complex of  $C_*(X_A^0)$  in which the only change is that  $C_0(X \text{rel}_0 A) = 0$ ; thus all elements of  $C_1(X \text{rel}_0 A)$  are cycles, and of course all generating elements of  $C_n(X \text{rel}_0 A)$  map vertices of  $I^n$  into  $A$ .

We write  $H_*(X \text{rel}_0 A)$  for the homology of this chain complex.  $\square$

**Theorem 5.6** *Let  $A$  be a subspace of the space  $X$ . Then a Hurewicz morphism*

$$\omega: \pi_1(X, A) \rightarrow H_1(X \text{rel}_0 A)$$

*is defined and induces an isomorphism*

$$\omega': \pi_1(X, A)^{\text{totab}} \rightarrow H_1(X \text{rel}_0 A).$$

**Proof** For each path class  $[f] \in \pi_1(X, A)$  the representative  $f$  determines a generator of  $C_1(X \text{rel}_0 A)$ . Differing choices of  $f$  yield homologous elements of  $C_1(X \text{rel}_0 A)$ , so this defines  $\omega$  as a function. If  $f \circ g$  is a composite of paths with vertices in  $A$  then the diagram

$$\begin{array}{ccc}
 & f \circ g & \\
 f \downarrow & \square & \downarrow 1 \\
 & g & 
 \end{array} \tag{10}$$

extends to a map of  $I^2 \rightarrow X$  with vertices mapped to  $A$  whose boundary shows that  $\omega$  is a morphism to  $H_1(X \text{rel}_0 A)$ . It hence defines  $\omega': \pi_1(X, A)^{\text{totab}} \rightarrow H_1(X \text{rel}_0 A)$ .

Now  $C_1(X \text{rel}_0 A)$  is free abelian on the non degenerate paths  $f: I \rightarrow X$  with vertices in  $A$ . So a morphism  $\eta: C_1(X \text{rel}_0 A) \rightarrow \pi_1(X, A)^{\text{ab}}$  is defined by sending  $f$  to its class in  $\pi_1(X, A)^{\text{ab}}$ . It is easy to check that  $\eta \partial_2 = 0$ , so that  $\eta$  defines a morphism

$$H_1(X \text{rel}_0 A) \rightarrow \pi_1(X, A)^{\text{totab}},$$

and that  $\eta$  is inverse to  $\omega'$ . □

Next we relate  $H_*(X \text{rel}_0 A)$  to the standard relative homology. The result we need generalises the classical case, when  $X$  is path connected and  $A$  consists of a single point, [Mas80, Lemma 7.2].

**Proposition 5.7** *If  $A$  meets each path component of  $X$ , then the inclusion*

$$C_*(X_A^0) \rightarrow C_*(X)$$

*is a chain equivalence.*

**Proof** This is an immediate consequence of Corollary 5.4. □

We say  $C_*(A)$  is *concentrated in dimension 0* if  $C_i(A) = 0$  for  $i > 0$ . This occurs for example if  $A$  is totally path disconnected, and so if  $A$  is discrete.

**Theorem 5.8 (Relative Hurewicz Theorem: dimension 1)** *If  $A$  is totally path disconnected and meets each path component of  $X$  then the natural morphism*

$$\pi_1(X, A)^{\text{totab}} \rightarrow H_1(X, A)$$

*is an isomorphism.*

**Proof** We define  $A_*$  to be the constant filtered space with value  $A$ . So we regard  $A_*$  as a sub-filtered space of  $X_A^0$ .

We consider the morphism of exact sequences of chain complexes

$$\begin{array}{ccccccc}
0 & \longrightarrow & C_*(A) & \longrightarrow & C_*(X) & \longrightarrow & C_*(X, A) \longrightarrow 0 \\
& & \downarrow = & & \downarrow i & & \downarrow j \\
0 & \longrightarrow & C_*(A_*) & \longrightarrow & C_*(X_A^0) & \longrightarrow & C_*(X_A^0, A_*) \longrightarrow 0
\end{array} \tag{11}$$

where classically the first sequence defines relative homology  $H_*(X, A)$ , and the second sequence defines  $H_*(X_A^0, A_*)$ . Under our assumptions, the morphism  $i$  is a homotopy equivalence and hence so also is  $j$  (since all the chain complexes are free in each dimension).

Our assumption that  $A$  is totally path disconnected implies that  $C_i(A) = 0$  for  $i > 0$ . This implies that  $C_*(X_A^0, A_*) \cong C_*(X_{\text{rel}0}A)$ . So the Theorem follows from Theorem 5.6 and Proposition 5.7.  $\square$

**Theorem 5.9** *Let  $p : \widehat{X} \rightarrow X$  be a covering map of connected spaces determined by the normal subgroup  $N$  of  $F = \pi_1(X, x)$ , and let  $\phi : F \rightarrow G = F/N$  be the quotient morphism. Let  $\widetilde{X}_0 = p^{-1}(x)$  so that there is a bijection*

$$\widetilde{X}_0 \cong G = F/N.$$

*Let  $\widehat{F}$  be as in Theorem 4.1. Then  $\widehat{F} \cong \pi_1(\widehat{X}, \widetilde{X}_0)$  and so there is an isomorphism*

$$\widehat{F}^{\text{totab}} \cong H_1(\widehat{X}, \widetilde{X}_0).$$

*Hence there is a universal  $\phi$ -derivation  $\pi_1(X, x) \rightarrow H_1(\widehat{X}, \widetilde{X}_0)$ .*

**Proof** The covering morphism of groupoids  $\pi_1(\widehat{X}, \widetilde{X}_0) \rightarrow F = \pi_1(X, x)$  is clearly isomorphic to  $\widehat{F} \rightarrow F$  since they are both determined by the normal subgroup  $N$  of  $F$ . But now Theorem 5.8 gives the result.  $\square$

**Remark 5.10** This description of  $\pi_1(X, x) \rightarrow H_1(\widehat{X}, \widetilde{X}_0)$  as a universal  $\phi$ -derivation is essentially the result of Crowell [Cro71, Section 5]. It also gives the case  $i = 1$  of [BH90, Proposition 5.2] which relates the fundamental crossed complex of certain filtered spaces to the chains with operators defined by universal covers. That Proposition generalises work in [Whi49, Section 11]; Whitehead's use of the chains of the universal cover in this and a number of papers was strongly influenced by Reidemeister's paper [Rei34]. That work developed also into work of Eilenberg–Mac Lane on homology of spaces with operators, [Eil47, EML49], as well as into Fox's work on the free differential calculus [Fox53].  $\square$

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