Chapter 9

Computation of the fundamental groupoid

In this chapter we compute the fundamental groupoid of some useful adjunction spaces, and hence of cell complexes, by applying the methods developed in chapter 8. We then develop some geometric applications, to knots, the Phragmen-Brouwer property, and the Jordan Curve Theorem.

9.1 The Van Kampen theorem for adjunction spaces

In Section 6.7 we proved a van Kampen theorem for the fundamental groupoid πXA when X is given as a union of open subsets.

Suppose given an adjunction space $W\ {}_f\!\sqcup\ Z$ as in the pushout square

$$\begin{array}{ccc} Y & \xrightarrow{\ f\ } & W \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle \bar{i}} \\ Z & \xrightarrow[\ \bar{f}\ ]{} & W\ {}_f\!\sqcup\ Z. \end{array} \qquad (9.1.1)$$

Our object is to determine the groupoid

$$\pi(W\ {}_f\!\sqcup\ Z)B$$

for certain (useful) B.

In chapter 7, we studied the homotopy type of $W\,{}_f\!\sqcup Z$ and showed its dependence on the homotopy types of W and (Z, Y) if (Z, Y) is cofibred. This is a local condition on Y in Z. To determine the fundamental groupoid $\pi(W\,{}_f\!\sqcup Z)$ as a pushout, we also need some local conditions—these conditions are essentially in dimensions 0 and 1, and are described in terms of the natural map

$$p : M(f) \cup Z \to W\,{}_f\!\sqcup Z$$

and its induced morphism of fundamental groupoids.

Suppose that C is a subset of Z representative in Z and in Y, that D is a subset of W representative in W, and that $f[C] \subseteq D$. Let

$$g = f \mid C \cap Y, D, \qquad B = D\,{}_g\!\sqcup C.$$

Under these conditions we have:

9.1.2 *(The Van Kampen theorem) The following square*

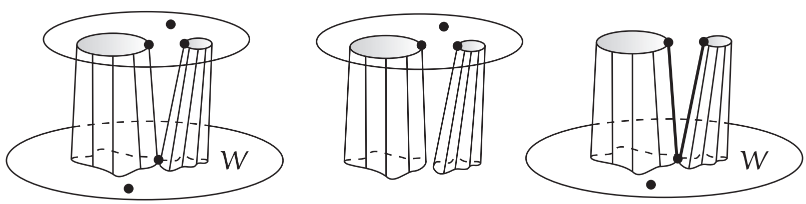
$$\begin{array}{ccc} \pi YC & \xrightarrow{\ f\ } & \pi WD \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle \bar{i}} \\ \pi ZC & \xrightarrow[\ \bar{f}\ ]{} & \pi(W\,{}_f\!\sqcup Z)B \end{array} \qquad (9.1.3)$$

is a pushout if and only if the morphism

$$p : \pi(M(f) \cup Z)A \to \pi(W\,{}_f\!\sqcup Z)B$$

in which $A = D \cup C$*, is a homotopy equivalence of groupoids. In particular,* (9.1.3) *is a pushout if* (Z, Y) *is cofibred.*

Proof Let $X = M(f) \cup Z$, $X_2 = X \setminus W$, $X_1 = M(f)$, $X_0 = X_1 \cap X_2$.



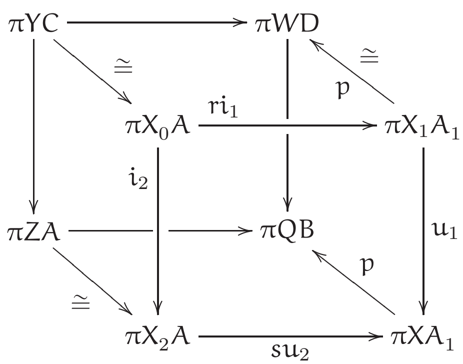
X $\qquad$ X_2 $\qquad$ X_1

Fig. 9.1

The interiors of X_1, X_2 cover X and so we are in the position to apply 6.7.4 to give us a pushout isomorphic to (9.1.3).

Let $A' = D$, so that in the notation of 6.7.4, $A_1 = D \cup (C \setminus Y)$. Then $p : M(f) \cup Z \to W\,{}_f\!\sqcup\, Z$ maps A_1 bijectively onto B. Further A', and hence also A_1, is representative in X_1. Indeed, A_1 meets each path component of W because D is representative in W. Also $C \cap Y$ is representative in Y and each point c of $C \cap Y$ can be joined by the path down the mapping cylinder of fc, which belongs to D; this path is shown as a thick line in X_1 in Fig. 9.1. Notice also that if θc is the class in $\pi X_1 A$ of this path, then $p(\theta c)$ is the identity at fc in πW.

Consider the following diagram in which (i) $Q = W\,{}_f\!\sqcup\, Z$, (ii) the front square is the pushout determined by A_1 and the above elements θc as in 6.7.4, (iii) the back square is (9.1.3).



(9.1.4)

The left-hand square is induced by inclusions and so is commutative. The right-hand square is induced by p and its restrictions, so the right-hand square is commutative. The commutativity of the top and bottom squares is a consequence of $p(\theta c) = 1$ ($c \in C \cap Y$). Thus (9.1.4) is commutative.

Each morphism marked $\cong$ is induced by a homotopy equivalence and is bijective on objects. Therefore these morphisms are isomorphisms. Hence, each of the following statements is equivalent to its successor: (a) (9.1.3) is a pushout, (b) (9.1.4) determines an isomorphism of its front square to its back square, (c) $p : \pi X A_1 \to \pi Q B$ is an isomorphism.

However, the last morphism is bijective on objects so (c) is equivalent to (d) $p : \pi X A_1 \to \pi Q B$ is a homotopy equivalence. Since $\pi X A_1$ is a deformation retract of $\pi X A$, (d) itself is equivalent to (e) $p : \pi X A \to \pi Q B$ is a homotopy equivalence.

This proves the main part of 9.1.2. The last statement of 9.1.2 follows from 7.5.4. □

9.1.2 *(Corollary 1) Let* W, Z *be closed in* $W \cup Z$*, let* $(Z, W \cap Z)$ *be cofibred and let* B *be a set representative in* $W \cap Z,\ W,\ Z$*. Then the square of morphisms induced by inclusions*

$$\begin{array}{ccc} \pi(W \cap Z)B & \longrightarrow & \pi W B \\ \downarrow & & \downarrow \\ \pi Z B & \longrightarrow & \pi(W \cup Z)B \end{array}$$

is a pushout.

Proof This is a consequence of 9.1.2 with $f : Y \to W$ the inclusion. □

This result is, of course, similar to 6.7.2, and is in many cases more convenient to use than the earlier result.

9.1.2 *(Corollary 2) Suppose the assumptions of* 9.1.2 (Corollary 1) *hold and also* $\pi(W \cap Z)B$ *is discrete. Then* $\pi(W \cup Z)B$ *is isomorphic to the free product of groupoids*

$$\pi Z B * \pi W B.$$

Proof From the pushout square of 9.1.2 (Corollary 1) it is easy to deduce the morphism $\pi Z B \sqcup \pi W B \to \pi(W \cup Z)B$ determined by the two inclusions of Z, W into $W \cup Z$ is a universal morphism. □

Remark Even this corollary is false without some local assumptions on Y in Z (or in W). For example, let H be the subspace of $\mathbb{R}^2$ which is the union of all circles centre $(1/n, 0)$ for n a positive integer—this space has been called the 'Hawaiian earring'. Let $0 = (0, 0)$ be the base point of H. The space CH is contractible and so the group $\pi(CH, 0)$ is trivial. However, [Gri54], [Gri56] has shown that $\pi(CH \vee CH, 0)$ is non-trivial and in fact can be generated only by an uncountable number of elements. Again, the fundamental group of $H \vee H$ is not the free product $\pi(H, 0) * \pi(H, 0)$.

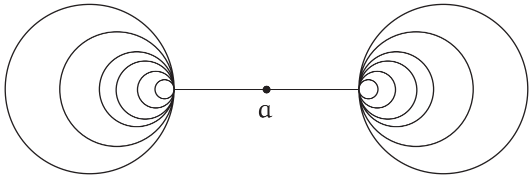


Fig. 9.2

However the space X of Fig. 9.2 formed by joining two Hawaiian earrings together does have has its fundamental group isomorphic to $\pi(H, 0) * \pi(H, 0)$—this is easy to prove from 9.1.2 (Corollary 2) by taking W, Z to be left- and right-hand halves of X meeting in $\{a\}$. In this case, the obvious map $X \to H \vee H$ induces a morphism $\pi(X, a) \to \pi(H \vee H, 0)$ which is injective but not surjective (the proof of this statement is not easy—cf. [Gri54] and [MM86]).

Suppose now that we are in the situation of 9.1.2, that (Z, Y) is cofibred, that $C \subseteq Y$ and $B = D = f[C]$.

9.1.2 *(Corollary 3) If further πYC, πWD are discrete then*

$$\bar{f} : \pi ZC \to \pi(W\ {}_f\sqcup\ Z)D$$

is a universal morphism.

Proof This follows from 9.1.2 and the definition of universal morphism. □

We can derive a number of useful results from this. For example, if D consists of a single point d (and the other assumptions of 9.1.2 (Corollary 3) hold) then the fundamental group $\pi(W\ {}_f\sqcup\ Z, d)$ is isomorphic to $U(\pi ZC)$, the universal group of πZC. In particular, if Z is path-connected and $c_0 \in C$, then

$$\pi(W\ {}_f\sqcup\ Z, d) \cong \pi(Z, c_0) * F$$

where F is a free group with one generator for each element of C other than c_0.

We now derive the fundamental group of a cell complex, first dealing with the 1-dimensional case.

9.1.5 *If K is a connected cell complex and $v \in K^0$, then the groupoid $\pi K^1 K^0$ is a free groupoid and the fundamental group $\pi(K^1, v)$ is a free group on $r_1 - r_0 + 1$ generators where r_n is the number of n-cells of K, $n = 0, 1$.*

Proof K^1 is obtained by adjoining 1-cells to K^0, that is,

$$K^1 = K^0\ {}_f\sqcup\ (\Lambda \times \mathbb{E}^1)$$

where Λ is a discrete set and $f : \Lambda \times \mathbb{S}^0 \to K^0$ is the attaching map. Let $C = \Lambda \times \mathbb{S}^0$; since K^1 is connected, $f[C] = K^0$. Since $\pi(\Lambda \times \mathbb{S}^0)C$ and $\pi K^0 K^0$ are discrete groupoids (and also $(\Lambda \times \mathbb{E}^1, \Lambda \times \mathbb{S}^0)$ is cofibred) the morphism

$$\bar{f} : \pi(\Lambda \times \mathbb{E}^1)C \to \pi K^1 K^0$$

is a universal morphism. But $\pi(\Lambda \times \mathbb{E}^1)C$ is isomorphic to $\Lambda \times \mathbf{I}$. So the result follows from the discussion of free groupoids in section 8.2. □

Notice that 9.1.5 also gives the generators of $\pi K^1 K^0$ as follows. For each λ in Λ let ι_λ denote the unique path class in $\pi(\lambda \times \mathbb{E}^1)C$ from $(\lambda, -1)$ to $(\lambda, +1)$. Then the generators of $\pi K^1 K^0$ are the elements $\bar{f}(\iota_\lambda)$, λ in Λ; thus if v_0, v_1 are vertices of K joined by a 1-cell, then the path class in $\pi K(v_0, v_1)$ determined by the characteristic map of this 1-cell is one of these generators.

9.1.5 *(Corollary 1) The fundamental group of the circle,* $\pi(\mathbb{S}^1, 1)$ *is isomorphic to* $\mathbb{Z}$*, with generator the class of the path*

$$\begin{aligned} \mathbb{I} &\to \mathbb{S}^1 \\ t &\mapsto e^{2\pi i t}. \end{aligned}$$

Proof This is immediate from 9.1.2 and the previous remark, since the given path is a characteristic map for the 1-cell of $\mathbb{S}^1$. □

We now show that $\pi(K^2, v)$ is isomorphic to $\pi(K^1, v)$ with relations for each 2-cell. Let us suppose

$$K^2 = K^1 \,{}_g\!\sqcup\, (M \times \mathbb{E}^2)$$

where $g : M \times \mathbb{S}^1 \to K^1$. Suppose also that K^1 is connected. For each m in M, let $v_m = g(m, e)$ where $e = (1, 0)$ and let

$$\rho_m = g(\iota_m) \in \pi(K^1, v_m)$$

where ι_m is a generator of the fundamental group of $M \times \mathbb{S}^1$ at (m, e). Let v be an element of K^1 and let a_m be an assigned element of $\pi K^1(v, v_m)$ (with $a_m = 1$ if perchance $v_m = v$).

9.1.6 *The fundamental group* $\pi(K^2, v)$ *is isomorphic to the free group* $\pi(K^1, v)$ *with the relations*

$$a_m^{-1} \rho_m a_m = 1, \quad m \in M.$$

Proof We first show that if $V = \{v_m : m \in M\} \cup \{v\}$ then $\pi K^2 V$ is the groupoid $\pi K^1 V$ with the relations $\rho_m = 1$, $m \in M$. Let $C = M \times \{e\}$. We have a pushout square

$$\begin{array}{ccc} \pi(M \times \mathbb{S}^1)C & \xrightarrow{\;g\;} & \pi K^1 V \\ \Big\downarrow{\scriptstyle i} & & \Big\downarrow{\scriptstyle \bar{i}} \\ \pi(M \times \mathbb{E}^2)C & \xrightarrow[\;\bar{g}\;]{} & \pi K^2 V. \end{array}$$

Suppose $f : \pi K^1 V \to F$ is any morphism such that $f\rho_m = 1$, $m \in M$. Then $\operatorname{Im}(fg)$ is discrete. Since $\pi(M \times \mathbb{E}^2)C$ is a discrete groupoid on C, f defines a morphism $\bar{f} : \pi(M \times \mathbb{E}^2)C \to F$ such that $\bar{f}i = fg$. So there is a unique morphism $f' : \pi K^2 V \to F$ such that $f'\bar{i} = f$, $f'\bar{g} = \bar{f}$. The last condition is redundant, since $\pi(M \times \mathbb{E}^2)C$ is discrete and so $\bar{g}$ is determined by $\operatorname{Ob}(\bar{g}) = \operatorname{Ob}(g) : C \to V$, a surjective function.

This proves that $\pi K^2 V$ is $\pi K^1 V$ with relations $\rho_m = 1$, $m \in M$. The conclusion of 9.1.6 follows from 8.3.3. □

9.1.7 *If* K *is a cell complex and* A *a subset of* K^2*, then the inclusion* $K^2 \to K$ *induces an isomorphism* $\pi K^2 A \to \pi K A$.

Proof We first prove that $\mathbb{S}^n$ is simply-connected for $n > 1$. Let e be a point of $\mathbb{S}^{n-1} = E^n_+ \cap E^n_-$. By 9.1.2 (Corollary 1) we have a pushout square

$$\begin{array}{ccc} \pi(\mathbb{S}^{n-1}, e) & \longrightarrow & \pi(E^n_+, e) \\ \Big\downarrow & & \Big\downarrow \\ \pi(E^n_-, e) & \longrightarrow & \pi(\mathbb{S}^n, e). \end{array}$$

But E^n_+, E^n_- are homeomorphic to $\mathbb{E}^n$ and so are simply-connected. Hence $\pi(E^n_+, e), \pi(E^n_-, e)$ are trivial groups and therefore $\pi(\mathbb{S}^n, e)$ is trivial.

Now consider any adjunction space $W \,{}_f\!\sqcup\, \mathbb{E}^{n+1}$ where $f : \mathbb{S}^n \to W$ and $n > 1$. Let $e \in \mathbb{S}^n$ and suppose W is path-connected. There is a pushout square

$$\begin{array}{ccc} \pi(\mathbb{S}^n, e) & \xrightarrow{\ f\ } & \pi(W, fe) \\ \Big\downarrow i & & \Big\downarrow \bar{i} \\ \pi(\mathbb{E}^{n+1}, e) & \xrightarrow[\bar{f}]{} & \pi(W\ {}_f\!\sqcup\ \mathbb{E}^{n+1}, fe). \end{array}$$

Since the two left-hand groups are trivial, $\bar{i}$ is an isomorphism. □

In order to compute the fundamental group of spaces, it is clearly necessary to compute maps $\pi(\mathbb{S}^1, e) \to \pi(K^1, fe)$. The following result is crucial.

9.1.8 *Let* $f : \mathbb{S}^1 \to \mathbb{S}^1$ *be the map* $z \mapsto z^n$, n *an integer. Then the induced morphism* $f : \pi(\mathbb{S}^1, e) \to \pi(\mathbb{S}^1, e)$ *of additive groups is multiplication by* n.

Proof The result is clearly true if $n = 0$, since f is then constant, or if $n = 1$, since f is then the identity. Suppose $n > 1$; let w be the complex number $e^{2\pi i/n}$ and let $w^r = e^{2\pi ir/n}$, $r = 0, 1, \ldots, n-1$. Let X_r be the subset of $\mathbb{S}^1$ of points $e^{2\pi i\theta}$, $r/n \leqslant \theta \leqslant (r+1)/n$, let C_r consist solely of w^r, w^{r+1} and let $C = \{w^r : 0 \leqslant r < n\}$. Since X_r is simply-connected there is a unique element $\imath_r$ in $\pi X_r(w^r, w^{r+1})$. The morphism $\pi X_r C_r \to \pi\mathbb{S}^1 C$ induced by inclusion is injective and so we regard $\pi X_r C_r$ as a subgroupoid of $\pi\mathbb{S}^1 C$. A generator a of $\pi(\mathbb{S}^1, e)$ where $e = w^0$, is then given by

$$a = \imath_{n-1} + \cdots + \imath_0.$$

The map $f : \mathbb{S}^1 \to \mathbb{S}^1$ determines by restriction $f' : X_r \to \mathbb{S}^1$; clearly $f'\imath_r = a$. Therefore,

$$fa = f(\imath_{n-1} + \cdots + \imath_0) = a + \cdots + a = na.$$

If $n < 0$, let $m = -n$. The $z \mapsto z^n$ is the composite of $g : z \mapsto z^{-1}$ and $z \mapsto z^m$. But if $b : \mathbb{I} \to \mathbb{S}^1$ is the path $t \mapsto e^{2\pi it}$, then gb is $t \mapsto e^{-2\pi it}$, that is, $gb = -b$. Hence, in $\pi(\mathbb{S}^1, e)$, $ga - fa$; therefore $fa = -ma = na$. □

If $f : \mathbb{S}^1 \to \mathbb{S}^1$ is a map such that $f : \pi(\mathbb{S}^1, 1) \to \pi(\mathbb{S}^1, 1)$ is multiplication by n, then we say f is of *degree* n.

EXAMPLES

1. Let $K = \mathbb{S}^1 \vee \cdots \vee \mathbb{S}^1$ be a wedge of n circles, with the cell structure $e^0 \cup e^1_1 \cup \cdots \cup e^1_n$. Let v be the vertex of K. Then $\pi(K, v)$ is a free group on n-generators, the generators being the classes of the loops which pass once round one of the circles.

2. The fundamental group of the real projective plane $P^2(\mathbb{R})$ and the real projective n-space $P^n(\mathbb{R})$ $(n > 1)$ are the same, by 9.1.7 and the fact that $P^2(\mathbb{R})$ can be identified with the 2-skeleton of $P^n(\mathbb{R})$. Also, $P^2(\mathbb{R}) = \mathbb{S}^1 {}_f\sqcup \mathbb{E}^2$ where $f : \mathbb{S}^1 \to \mathbb{S}^1$ is of degree 2 [Section 5.3]. It follows that the fundamental group of $P^2(\mathbb{R})$ is the group $\mathbb{Z}/2\mathbb{Z} = \mathbb{Z}_2$.
3. We can also state that the fundamental groups of $\mathbb{S}^2$ and $\mathbb{S}^1 \times \mathbb{S}^1$ are 0 and $\mathbb{Z} \times \mathbb{Z}$ respectively. It follows that no two of the spaces $\mathbb{S}^2$, $\mathbb{S}^1 \times \mathbb{S}^1$, $P^n(\mathbb{R})$ are of the same homotopy type; *a fortiori*, no two of these spaces are homeomorphic.
4. The Klein bottle has a cell structure $K = e^0 \cup e^1_1 \cup e^1_2 \cup e^2$. From Fig. 4.4, p. 98, it is clear that, if $\{a, b\}$ is a set of generators of $\pi(K^1, v)$ as given in 9.1.5, then the relation determined by the 2-cell of K is $abab^{-1}$. Thus $\pi(K, v)$ is a free group on two generators a, b with the relation $abab^{-1} = 1$.

It is a simple consequence of 9.1.2 that if we form a pushout of spaces $Q = B {}_f\sqcup (X \times \mathbb{I})$ by attaching a cylinder $X \times \mathbb{I}$ to B by means of a map $f : X \times \dot{\mathbb{I}} \to B$ then the fundamental groupoid of Q, on an appropriate set, may be described as a pushout of groupoids in a manner analogous to (8.4.2). We leave the reader to describe this precisely.

The last result on HNN-extensions, and our earlier proofs that that the fundamental group of a circle is infinite cyclic, show that some groups are well described as constructed from groupoids. On the other hand, it is sometimes convenient to regard groups as object groups of groupoids. As an example, consider the trefoil group $\mathrm{Tr} = \mathrm{gp}\langle x, y \mid x^3y^{-2}\rangle$. This is known to be an infinite group, but from this viewpoint it is not so easy to find a normal form for its elements.

A different way into its structure comes from seeing the trefoil group as a fundamental group of a cell complex given as a double mapping cylinder. Let the unit circle $\mathbb{S}^1$ have base point e, say, and consider the double mapping cylinder $M = M(3, 2)$ of the maps $\mathbb{S}^1 \to \mathbb{S}^1$ given by $z \mapsto z^3$, $z \mapsto z^2$ respectively. This space M contains two copies of $\mathbb{S}^1$ with base points e_3, e_2, say. Let E be the set of these base points. Then the fundamental groupoid $\widehat{T} = \pi(M, E)$ has a presentation with three generators $\hat{x} \in \widehat{T}(e_3)$, $\hat{y} \in \widehat{T}(e_2)$, $w \in \widehat{T}(e_3, e_2)$ with the relation $w\hat{x}^3 = \hat{y}^2 w$. An advantage of this 'Trefoil groupoid' $\widehat{T}$ over the Trefoil group Tr is that the generator w acts as a kind of 'separator' of the generators $\hat{x}, \hat{y}$ for any well defined word in these generators not containing consecutive symbols u, u^{-1}, for $u = x, y, w$ or their inverses. For this reason, it is easy to give a normal form for words, and we leave this as an exercise for the reader.

Notice also that the double mapping cylinder M as above is a cell complex. By contrast, the pushout P of the two maps $\mathbb{S}^1 \to \mathbb{S}^1$ as above is not even Hausdorff, and it is not clear what might be its fundamental groupoid.

By contrast, the groupoid $\widehat{T}$ is easily understood: it is the double mapping cylinder in the category of groupoids of the two morphisms $\mathbb{Z} \to \mathbb{Z}$ given by multiplication by 2 and by 3 respectively.

We now give an intuitive account of the computation of the fundamental group of the complement of a graph embedded in Euclidean space $\mathbb{R}^3$. This computation is important in knot theory.

By a *graph* in $\mathbb{R}^3$ we mean a cell complex K of dimension 1 which is embedded in $\mathbb{R}^3$. Such a graph is usually represented by a diagram with vertices, overpasses, edges and crossings as in Fig. 9.3.

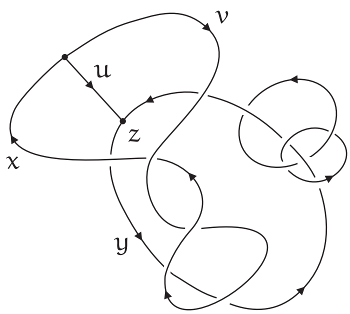


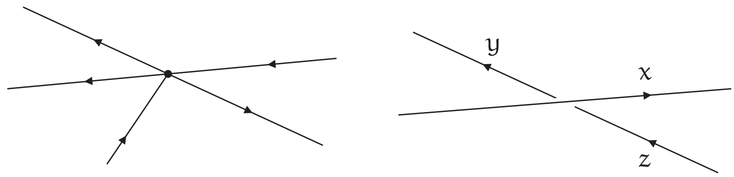
Fig. 9.3

In order to avoid *wildness* problems, for example edges which wind infinitely often round others (see [FA49]) we suppose given a 1-dimensional complex K and an embedding $i : K \times \mathbb{E}^2 \to \mathbb{R}^3$. That is, the edges of the graph are taken to have a certain thickness.

The diagram D of the embedding can be regarded as the projection of $i[K \times \mathbb{E}^2]$ onto $\mathbb{R}^2$ by the mapping $(x, y, z) \mapsto (x, y)$, and it is supposed that the embedding is arranged so that the diagram has edges crossing only at double points. This conforms with our picture. The diagram allows us to divide the embedded graph into *vertices*, namely the images of $v \times \mathbb{E}^2$ where v is a vertex of K at which more than two edges meet, and *overpasses*, namely the edges between vertices, between crossings, or between vertices and crossings. Thus the overpasses are labelled by letters in Fig. 9.3. We also orient the graph by choosing a direction for each overpass as shown.

9.1.9 *Under the above circumstances, the fundamental group* $\pi(\mathbb{R}^3 \setminus i[K \times \mathbb{E}^2], p)$ *has a presentation with a generator for each overpass and relations of two types:*
(a) *at each vertex* v *there is a relation* $x_1^{\varepsilon_1} x_2^{\varepsilon_2} \dots x_r^{\varepsilon_r} = 1$*, where the* x_i *are the edges at the vertex and the sign* ε_i *is* $+1$ *if the arrow for* x_i *points towards the vertex, and* -1 *otherwise;*

(b) *at each crossing with overpass* x *crossing* y *and* z *as shown in* Fig. 9.4 (ii)*, there is a relation* $y = xzx^{-1}$.



(i) (ii)

Fig. 9.4

A proof goes roughly as follows. Imagine the graph $i[K \times \mathbb{E}^2]$ as part of a slice $S = \mathbb{R}^2 \times [-\varepsilon, \varepsilon]$ of $\mathbb{R}^3$, while S is regarded as the intersection of two half spaces $H_+ = \mathbb{R}^2 \times [-\varepsilon, \rightarrow[$ and $H_- = \mathbb{R}^2 \times]\leftarrow, \varepsilon]$. Surround each vertex v by a solid ball E_v and each crossing c by a solid ball D_c as in Fig. 9.5.

E_v D_c

Fig. 9.5

Let X be the union of $i[K \times \mathbb{E}^2]$ and all the balls E_v and D_c. Arrange these so that $H_+ \setminus X$ and $H_- \setminus X$ intersect in a non connected space with one component for each region into which the diagram of the graph divides the plane. By 9.1.8, the fundamental group $\mathbb{R}^3 \setminus X$ is a free group with one generator for each edge e_i of the diagram between crossings or vertices. Now replace the balls E_v and D_c with their intersection with $i[K \times \mathbb{E}^2]$ excised. It is easy to see that E_v contributes a relation as given at each vertex. The balls D_c contribute two relations. One of these 'continues' an overpass while the other is the relation we want; that is in the situation shown in Fig. 9.6,

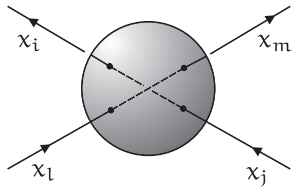


Fig. 9.6

the relations are

$$x_l = x_m, \quad x_i = x_l x_j x_k^{-1}.$$

Another way of seeing the intuitive basis of these relations is given in Fig. 9.7. The thick lines denote parts of the embedded graph, and the thin lines denote various positions of a deformation of part of a path. The reader is also urged to demonstrate these relations with string and wire models.

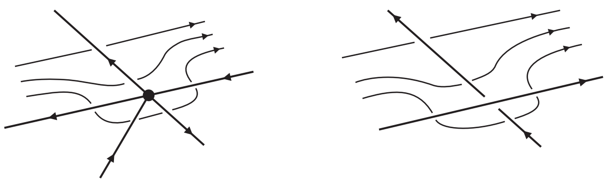


Fig. 9.7

As one application, for the pentoil shown in Fig. 9.8

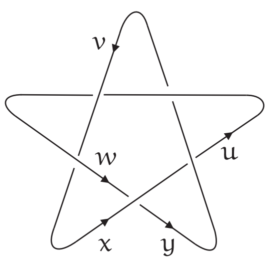


Fig. 9.8

we obtain a presentation for the fundamental group of the complement as having generators x, y, u, v, w with relations $w = xyx^{-1}$, $x = yuy^{-1}$, $y = uvu^{-1}$ and $v = wxw^{-1}$. By elimination of u, v, and w we may obtain the presentation with generators x and y and one relation

$$xyxyxy^{-1}x^{-1}y^{-1}x^{-1}y^{-1} = 1.$$

This relation corresponds to wrapping string around a part of knot model as shown in Fig. 9.9.

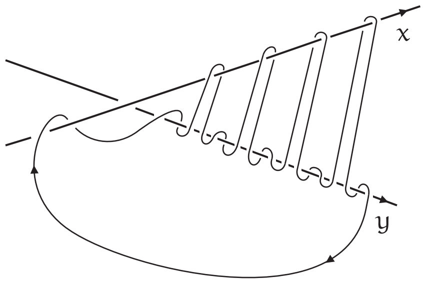


Fig. 9.9

If you wrap string around a model of the pentoil in precisely this way, and then tie the ends together, the loop will disentangle itself from the knot, thus demonstrating the calculation. (See [Bro88].)

For further information on knots and links, consult also [Kau87]. For relations of the fundamental group to the important area of configuration spaces (these are spaces of n distinct points in a space X) see also [Bir75].

EXERCISES

1. Prove that 9.1.7 is a consequence of the cellular approximation theorem.
2. Prove that the spaces $P^n(\mathbb{H})$ are simply-connected.
3. Prove that $\mathbb{S}^1$ is not a retract of $P^n(\mathbb{R})$, $n > 1$.
4. Let $X = Y \cup Z$ where Z, Y are path-connected and (X, Y) is cofibred. Let $a_0, a_1, \dots, a_n$ be points one in each path-component of $Y \cap Z$. Let i, j be the inclusions of $Y \cap Z$ into Y, Z respectively. Let $\alpha_r \in \pi Y(a_0, a_r)$, $\beta_r \in \pi Z(a_0, a_r)$, $r = 0, \dots, n$, with $\alpha_0 = 1$, $\beta_0 = 1$. Let F be a free group on elements γ_r, $r = 0, \dots, n$ with the relation $\gamma_0 = 1$. Prove that $\pi(X, a_0)$ is isomorphic to the free product of the groups $\pi(Y, a_0)$, $\pi(Z, a_0)$ and F with the relations

$$\alpha_r^{-1}(i\rho_r)\alpha_r = \gamma_r(\beta_r^{-1}(j\rho_r)\beta_r)\gamma_r^{-1}$$

for all $\rho_r \in \pi(Y \cap Z, a_r)$ and $r = 0, \dots, n$. [Here γ_r corresponds to the element $(u\beta_r^{-1})(v\alpha_r)$ of $\pi(X, a_0)$ where u, v are the inclusions of Y, Z respectively into X.]
5. Let K, L be 1-dimensional (finite) cell complexes. Prove that if $\varphi : \pi KK^0 \to \pi LL^0$ is any morphism, then there is a map $f : K \to L$ such that $\pi f = \varphi$. Prove also that if $f, g : K \to L$ are cellular maps such that $\pi f \simeq \pi g : \pi KK^0 \to \pi LL^0$, then f is homotopic to g.

6. Extend the results 9.1.5, 9.1.6, 9.1.7 to (infinite) CW-complexes. Prove that if G is any group, then there is a CW-complex K and a vertex x of K such that $\pi(K, x)$ is isomorphic to G. Deduce that if G is any groupoid then there is a CW-complex K such that πKK^0 is isomorphic to G. [You may assume that if G is any group then there is a free group F and a free subgroup R of F such that G is isomorphic to F/R.]
7. Prove that $\mathbb{R}^2$ and $\mathbb{R}^n$ for $n > 2$ are not homeomorphic.
8. Let $p : z \mapsto z^n + a_{n-1}z^{n-1} + \cdots + a_1 z + a_0$ and $q : z \mapsto z^n$ be polynomials with $a_i \in \mathbb{C}$. For $r > 0$ let $C_r = \{z \in \mathbb{R}^2 : |z| = r\}$. Prove that for r large enough, p and q restrict to homotopic maps $C_r \to \mathbb{R}^2 \setminus \{0\}$. Prove that for any $r > 0$ this restriction of q is essential and hence show that the polynomial p has a root. [This is known as the Fundamental Theorem of Algebra.]

9.2 The Jordan Curve Theorem

The Jordan Curve Theorem states that if C is a subset of the plane $\mathbb{R}^2$ such that C is homeomorphic to the circle $\mathbb{S}^1$, then $\mathbb{R}^2 \setminus C$ has exactly two components, one of them bounded and the other unbounded, and each with C as boundary. The set C is called a *simple closed curve in* $\mathbb{R}^2$. The bounded component of $\mathbb{R}^2 \setminus C$ is of course called the *inside* of the curve, and the unbounded component is called the *outside* of the curve.

This theorem is a classic instance of a result which at first sight seems intuitively obvious, but which needs some sophisticated machinery for its proof. In fact the theorem is not quite so obvious intuitively. Consider for example the computer generated simple closed curve in $\mathbb{R}^2$ shown in Fig. 9.10. How do you determine the inside and outside? (I am indebted to S. J. Abas for this diagram.)

Any method you choose for this curve might be defeated by a more complicated example. In any case, if you try and work out the inside and outside for this case, you begin to see the prospective complications of the problem.

In this section we shall use the final results of the last section to give a complete proof of the theorem; we also draw further consequences of the method. For this reason, we take what might seem a circuitous route to the theorem, by introducing a property which a space may or may not have.

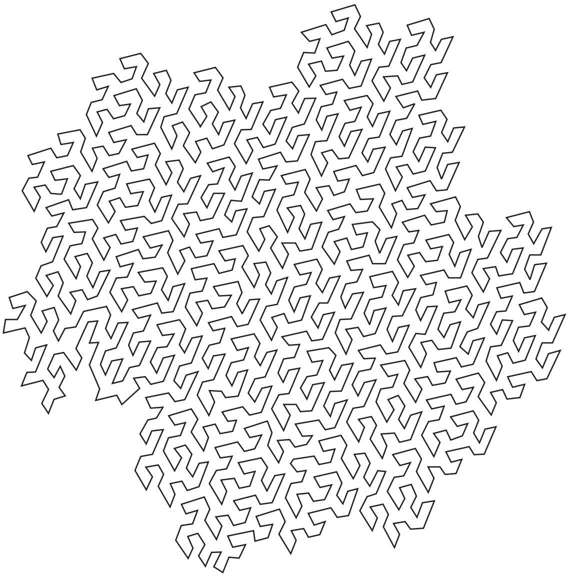


Fig. 9.10

A topological space X is said to *have the Phragmen-Brouwer property* (here abbreviated to PBP) if X is connected and the following holds: *if* D *and* E *are disjoint, closed subsets of* X*, and if* a *and* b *are points in* $X \setminus (D \cup E)$ *which lie in the same component of* $X \setminus D$ *and in the same component of* $X \setminus E$*, then* a *and* b *lie in the same component of* $X \setminus (D \cup E)$. To express this more succinctly, we say a subset D of a space X *separates* the points a and b if a and b lie in distinct components of $X \setminus D$. Thus the PBP is that: if D and E are disjoint closed subsets of X and a, b are points of X not in $D \cup E$ such that neither D nor E separate a and b, then $D \cup E$ does not separate a and b.

A standard example of a space not having the PBP is the circle $\mathbb{S}^1$, since we can take $D = \{+1\}$, $E = \{-1\}$, $a = i$, $b = -i$. This example is typical, as the next result shows. But first we remark that our criterion for the PBP will involve fundamental groups, that is will involve paths, and so we need to work with path-components rather than components. However, if X is locally path-connected, then components and path-components of open sets of X coincide, and so for these spaces we can replace in the PBP 'component' by 'path-component'. This explains the assumption of locally path-connected in the results that follow.

9.2.1 *Let* X *be a path-connected and locally path-connected space whose fun-*

damental group (at any point) does not have the integers $\mathbb{Z}$ as a retract. Then X has the PBP.

Proof Suppose X does not have the PBP. Then there are disjoint, closed subsets D and E of X and points a and b of $X \setminus (D \cup E)$ such that $D \cup E$ separates a and b but neither D nor E separates a and b. Let $X_1 = X \setminus D$, $X_2 = X \setminus E$, $X_0 = X \setminus (D \cup E) = X_1 \cap X_2$. Let J be a subset of X_0 such that $a, b \in J$ and J meets each path-component of X_0 in exactly one point. Since D and E do not separate a and b, there are elements $\alpha \in \pi X_1(a, b)$ and $\beta \in \pi X_2(a, b)$. Since X is path-connected, the set J is representative in X_0, X_1 and X_2. By 6.7.2 the following diagram of morphisms induced by inclusions is a pushout of groupoids:

$$\begin{array}{ccc} \pi X_0 J & \xrightarrow{i_1} & \pi X_1 J \\ {\scriptstyle i_2}\downarrow & & \downarrow{\scriptstyle u_1} \\ \pi X_2 J & \xrightarrow[u_2]{} & \pi X J. \end{array}$$

Since X_1 and X_2 are path-connected and J has more than one element, it follows from (9.1.9 (Corollary)) that πXJ has the integers $\mathbb{Z}$ as a retract. □

As an immediate application we obtain:

9.2.2 *The following spaces have the PBP: the sphere $\mathbb{S}^n$ for $n > 1$; $\mathbb{S}^2 \setminus \{a\}$ for $a \in \mathbb{S}^2$; $\mathbb{S}^n \setminus A$ if A is a finite set in $\mathbb{S}^n$ and $n > 2$.* □

In each of these cases the fundamental group is trivial.

An important step in our proof of the Jordan Curve Theorem is to show that if A is an *arc* in $\mathbb{S}^2$, that is a subspace of $\mathbb{S}^2$ homeomorphic to the unit interval $\mathbb{I}$, then the complement of A is path-connected. This follows from the following more general result.

9.2.3 *Let X be a path-connected and locally path-connected Hausdorff space such that for each x in X the space $X \setminus \{x\}$ has the PBP. Then any arc in X has path-connected complement.*

Proof Suppose A is an arc in X and $X \setminus A$ is not path-connected. Let a and b lie in distinct path-components of $X \setminus A$.

By choosing a homeomorphism $\mathbb{I} \to A$ we can speak unambiguously of the mid-point of A or of any subarc of A. Let x be the mid-point of A, so that A is the union of sub-arcs A' and A'' with intersection $\{x\}$. Since X is Hausdorff, the compact sets A' and A'' are closed in X. Hence $A' \setminus \{x\}$ and

$A'' \setminus \{x\}$ are disjoint and closed in $X \setminus \{x\}$. Also $A \setminus \{x\}$ separates a and b in $X \setminus \{x\}$ and so one at least of A', A'' separates a and b in $X \setminus \{x\}$. Write A_1 for one of A', A'' which does separate a and b. Then A_1 is also an arc in X.

In this way we can find by repeated bisection a sequence A_i, $i \geqslant 1$, of sub-arcs of A such that for all i the points a and b lie in distinct path-components of $X \setminus A_i$ and such that the intersection of the A_i for $i \geqslant 1$ is a single point, say y, of X.

Now $X \setminus \{y\}$ is path-connected, by definition of the PBP. Hence there is a path λ joining a to b in $X \setminus \{y\}$. But λ has compact image and hence lies in some $X \setminus A_i$. This is a contradiction. □

9.2.3 *(Corollary) The complement of any arc in $\mathbb{S}^n$ is path-connected.* □

In this theorem the case $n = 0$ is trivial, while the case $n = 1$ needs a special argument that the complement of any arc in $\mathbb{S}^1$ is an open arc. The case $n \geqslant 2$ follows from the above results.

We now prove one step along the way to the full Jordan Curve Theorem.

9.2.4 (The Jordan Separation Theorem) *The complement of a simple closed curve in $\mathbb{S}^2$ is not connected.*

Proof Let C be a simple closed curve in $\mathbb{S}^2$. Since C is compact and $\mathbb{S}^2$ is Hausdorff, C is closed, $\mathbb{S}^2 \setminus C$ is open, and so path-connectedness of $\mathbb{S}^2 \setminus C$ is equivalent to connectedness.

Write $C = A \cup B$ where A and B are arcs in C meeting only at a and b say. Let $U = \mathbb{S}^2 \setminus A$, $V = \mathbb{S}^2 \setminus B$, $W = U \cap V$, $X = U \cup V$. Then $W = \mathbb{S}^2 \setminus C$ and $X = \mathbb{S}^2 \setminus \{a, b\}$. Also X is path-connected, and, by 9.2.3 (Corollary), so also are U and V.

Let $x \in W$. Suppose that W is path-connected. By 6.7.2, the following diagram of morphisms induced by inclusion is a pushout of groups:

$$\begin{array}{ccc} \pi(W, x) & \longrightarrow & \pi(U, x) \\ \big\downarrow & & \big\downarrow {\scriptstyle i_*} \\ \pi(V, x) & \xrightarrow[j_*]{} & \pi(X, x). \end{array}$$

Now $\pi(X, x)$ is isomorphic to the group $\mathbb{Z}$ of integers. We derive a contradiction by proving that the morphisms i_* and j_* are trivial. We give the proof for i_*, as that for j_* is similar.

Let $f : \mathbb{S}^1 \to U$ be a map and let $g = if : \mathbb{S}^1 \to X$. Let γ be a parametrisation of A which sends 0 to b and 1 to a. Choose a homeomorphism

$h : \mathbb{S}^2 \setminus \{a\} \to \mathbb{R}^2$ which takes b to 0 and such that hg maps $\mathbb{S}^1$ into $\mathbb{R}^2 \setminus \{0\}$. Then $h\gamma(0) = 0$ and $\|h\gamma(t)\|$ tends to infinity as t tends to 1. Since the image of g is compact, there is an $r > 0$ such that $hg[\mathbb{S}^1]$ is contained in $B(0, r)$. Now there exists $0 < t_0 < 1$ such that the distance from 0 to $y = h\gamma(t_0)$ is $> r$. Define the path λ to be the part of $h\gamma$ reparametrised so that $\lambda(0) = 0$ and $\lambda(1) = y$.

Define $G : \mathbb{S}^1 \times \mathbb{I} \to \mathbb{R}^2$ by

$$G(z,t) = \begin{cases} hg(z) - \lambda(2t) & \text{if } 0 \leqslant t \leqslant \frac{1}{2}, \\ (2-2t)hg(z) - y & \text{if } \frac{1}{2} \leqslant t \leqslant 1. \end{cases}$$

Then G is well-defined. Also G never takes the value 0 (this explains the choices of λ and y). So G gives a homotopy in $\mathbb{R}^2 \setminus \{0\}$ from hg to the constant map at $-y$. So hg is inessential and hence g is inessential. This completes the proof that i_* is trivial. □

As we shall see, the Jordan Separation Theorem is used in the proof of the Jordan Curve Theorem.

9.2.5 (Jordan Curve Theorem) *If C is a simple closed curve in $\mathbb{S}^2$, then the complement of C has exactly two components, each with C as boundary.*

Proof As in the proof of 9.2.4, write C as the union of two arcs A and B meeting only at a and b say, and let $U = \mathbb{S}^2 \setminus A$, $V = \mathbb{S}^2 \setminus B$. Then U and V are path-connected and $X = U \cup V = \mathbb{S}^2 \setminus \{a, b\}$ has fundamental group isomorphic to $\mathbb{Z}$. Also $W = U \cap V = \mathbb{S}^2 \setminus C$ has at least two path-components, by 9.2.4.

If W has more than two path-components, then the fundamental group G of X contains a copy of the free group on two generators, by 9.1.9 (Corollary), and so G is non-abelian. This is a contradiction, since $G \cong \mathbb{Z}$. So W has exactly two path-components P and Q, say, and this proves the first part of 9.2.5.

Since C is closed in $\mathbb{S}^2$ and $\mathbb{S}^2$ is locally path-connected, the sets P and Q are open in $\mathbb{S}^2$. It follows that if $x \in \overline{P} \setminus P$ then $x \notin Q$, and hence $\overline{P} \setminus P$ is contained in C. So also is $\overline{Q} \setminus Q$, for similar reasons. We prove these sets are equal to C.

Let $x \in C$ and let N be a neighbourhood of x in $\mathbb{S}^2$. We prove N meets $\overline{P} \setminus P$. Since $\overline{P} \setminus P$ is closed and N is arbitrary, this proves that $x \in \overline{P} \setminus P$.

Write C in a possibly new way as a union of two arcs D and E intersecting in precisely two points and such that D is contained in $N \cap C$. Choose points p in P and q in Q. Since $\mathbb{S}^2 \setminus E$ is path-connected, there is a path λ joining p to q in $\mathbb{S}^2 \setminus E$. Then λ must meet D, since p and q lie in distinct path-components of $\mathbb{S}^2 \setminus E$. In fact if $s = \sup\{t \in \mathbb{I} : \lambda[0,t] \subseteq P\}$, then $\lambda(s) \in \overline{P} \setminus P$. It follows that N meets $\overline{P} \setminus P$.

So $\overline{P} \setminus P = C$ and similarly $\overline{Q} \setminus Q = C$. □

NOTES

There are many books containing a further discussion of this area. For more on the Phragmen-Brouwer property, see [Why42] and [Wil49]. Wilder lists five other properties which he shows for a connected and locally connected metric space are each equivalent to the PBP. The above proof of the Jordan Curve Theorem is adapted from [Mun75]. Because he does not have our Van Kampen theorem for non-connected spaces, he is forced into rather special covering space arguments to prove his replacements for 9.1.9 (Corollary) and for 9.2.1. As far as I am aware, 9.2.1 and 9.2.2 are not previously published.

A different kind of proof of the Jordan Curve Theorem is given in [Mae84]; this uses only the Brouwer Fixed Point Theorem and the Tietze Extension Theorem.

An important strengthening of the Jordan Curve Theorem is the Schoenflies Theorem: : *if* C *is a simple closed curve in* $\mathbb{S}^2$ *then each component of* $\mathbb{S}^2 \setminus C$ *is homeomorphic to* $\mathbb{R}^2$*; in fact, there is a homeomorphism* $h : \mathbb{R}^2 \to \mathbb{R}^2$ *which takes* C *to the standard circle* $\mathbb{S}^1$. For more information on this area, see for example [Moi77], [Bin83], [Sie05], and the web site `www.maths.ed.ac.uk/~aar/jordan/`.

For exercises in this area, see [Mun75].

An important part of algebraic topology deals with braids, links and mapping class groups [Bir75], but the results given in this book on groupoids have hardly been used in this area.