Introduction by Ronnie Brown

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The theory we describe in this book was developed over a long period, starting about 1965, and always with the aim of developing groupoid methods in homotopy theory of dimension greater than 1. Algebraic work made substantial progress in the early 1970s, in work with Chris Spencer, and in 1974 by Brown and Higgins. A substantial step forward in 1974 led us over the years into many fruitful areas of homotopy theory and what is now called higher dimensional algebra. We published detailed reports on all we found as the journey proceeded, but the overall picture of the theory is still not well known. So the aim of this book is to give a full, connected account of this work in one place, so that it can be more readily evaluated and used appropriately.

1 Structure of the subject

There are several features of the theory and so of our exposition which divert from standard practice in algebraic topology, but are essential for the full success of our methods.

1.1 Sets of base points: Enter groupoids

The notion of a space with base point is standard in algebraic topology and homotopy theory, but in many situations we are unsure which one to choose. One example is if $p : Y \to X$ is a covering map of spaces. Then X may have a chosen base point x, but it is not clear which base point to choose in the discrete space $p^{-1}(x)$. It makes sense then to take $p^{-1}(x)$ as a set of base points.

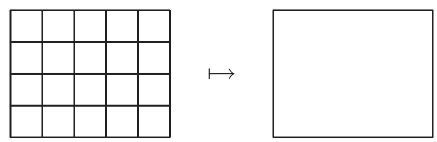
Choosing a set of base points according to the geometry of the situation has the implication that we deal with fundamental groupoids $\pi_1(X, X_0)$ on a set X_0 of base points rather than the family of fundamental groups $\pi_1(X, x),\ x \in X_0$. The intuitive idea is to consider X as a country with railway stations at the points of X_0; we then want to think of all the journeys *between* the stations and not just what is usually called 'change of base point', the somewhat bizarre concept of the set of return journeys from the individual stations, together with ways of moving from a return journey at one station to a return journey at another.

Sets of base points are used freely in van Kampen[1]type situations in [Bro06], when two connected open sets U, V have a disconnected intersection $U \cap V$. In such case it is sensible to choose a set X_0 of base points, say one point in each component of the intersection.

The method is to use a van Kampen type theorem to pass from topology to algebra by determining the fundamental groupoid $\pi_1(U \cup V, X_0)$ of a union, and then to use combinatorial groupoid methods to compute a particular fundamental group $\pi_1(U \cup V, x)$. This follows the principle of keeping track of structure for as long as is reasonable.

1.2 Groupoids in 2-dimensional homotopy theory

The successful use of groupoids in 1-dimensional homotopy theory in [Bro68] suggested the desirability of investigating the use of groupoids in higher homotopy theory. One aspect was to find a mathematics which allowed 'algebraic inverse to subdivision', in the sense that it could represent multiple compositions as in the following diagram:

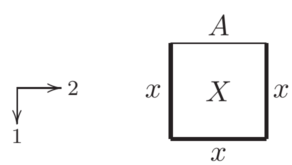


(multcomp)

in a manner analogous to the use of $(a_1, a_2, \dots, a_n) \mapsto a_1 a_2 \dots a_n$ in categories and groupoids, but in two dimensions. Note that going from right to left in the diagram is subdivision, a standard technique in mathematics.

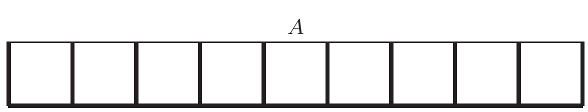
Traditional homotopy theory described the family $\pi_2(X, x)$ of homotopy groups, consisting of homotopy classes of maps $I^2 \to X$ which take the edges of the square I^2 to x, but this did not incorporate the groupoid idea, except under 'change of base point'.

Also considered were the relative homotopy groups $\pi_n(X, A, x)$ of a based pair (X, A, x) where $x \in A \subseteq X$. In dimension 2 the picture is as follows, where thick lines denote constant maps:



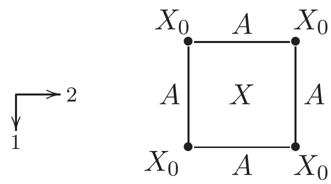
That is, we have homotopy classes of maps from the square I^2 to X which take the edge ∂_1^- to A, and the remaining three edges to the base point.

This definition involves choices, is unsymmetrical with respect to directions, and so is unaesthetic. The composition in $\pi_2(X, A, x)$ is the clear horizontal composition, and does give a group structure, but even large compositions are still 1-dimensional, i.e. in a line:



In 1974 Brown and Higgins found a new construction which we called $\rho_2(X, A, X_0)$ (published in [BH78]): it involves no such choices, and really does enable multiple compositions as wished for in

Diagram (multcomp). We considered homotopy classes rel vertices of maps $[0,1]^2 \to X$ which map edges to A and vertices to X_0:



Part of the geometric structure held by this construction is shown in the diagram:

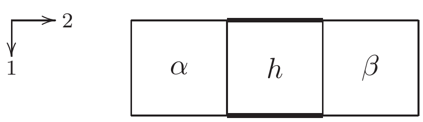
$$\rho_2(X, A, X_0) \substack{\rightarrow\\\rightarrow\\\rightarrow\\\rightarrow} \pi_1(A, X_0) \rightrightarrows X_0$$

where the arrows denote boundary maps.

A horizontal composition in $\rho_2(X, A, X_0)$ is given by

$$\langle\!\langle \alpha \rangle\!\rangle +_2 \langle\!\langle \beta \rangle\!\rangle = \langle\!\langle \alpha +_2 h +_2 \beta \rangle\!\rangle$$

as shown in the following diagram, where h is a homotopy rel end points in A between an edge of α and an edge of β, and thick lines show constant paths.



The proof that this composition is well defined on homotopy classes is not entirely trivial and is given in Chapter 6. With a similar vertical composition, we obtain the structure of *double groupoid*, which enables multiple compositions as asked for in Diagram (multcomp).

There is still more structure which can be given to ρ_2, namely that of 'connections', which we describe in Section 1.8.

1.3 Crossed modules

A surprise was that the investigation of double groupoids led back to a concept due to Henry Whitehead when investigating the properties of second relative homotopy groups, that of crossed module. Analogous ideas were developed independently by K. Reidemeister, the war having led to zero contact between mathematicians in Germany and the UK. Work with C.B. Spencer in 1971-73 led to the discovery of a close relation between double groupoids and crossed modules. This with the construction in the previous subsection led to a 2-dimensional van Kampen theorem, making possible some new computations of nonabelian second relative homotopy groups.

A *crossed module* is a morphism

$$\mu : M \to P$$

of groups together with an action of the group P on the right of the group M, written $(m, p) \mapsto m^p$, satisfying the two[2] rules:

CM1) $\mu(m^p) = p^{-1}(\mu m)p$;

CM2) $m^{-1}nm = n^{\mu m}$,

for all $p \in P,\ m, n \in M$. Algebraic examples of crossed modules include normal subgroups M of P; P-modules; the inner automorphism crossed module $M \to \operatorname{Aut} M$; and many others. There is the beginnings of a combinatorial and also computational crossed module theory.

The standard geometric example of crossed module is the boundary morphism of the second relative homotopy group

$$\partial : \pi_2(X, X_1, x) \to \pi_1(X_1, x)$$

where X_1 is a subspace of the topological space X and $x \in X_1$. Our 2-dimensional van Kampen theorem computes this crossed module in many useful conditions when X is a union of open sets.

Traditionally the focus has been on the second homotopy group. However Mac Lane and Whitehead had shown that crossed modules model pointed homotopy 2-types, so that the 2-dimensional van Kampen theorem allowed new computations of some homotopy 2-types.

Thus a difficult aim to compute a second homotopy group was reached by computing a larger structure, the homotopy 2-type, which also in principle determined the homotopy group not just as an abelian group but also as a module over the fundamental group.

For all these reasons, crossed modules are commonly seen as good candidates for 2-*dimensional groups*.

The algebra of crossed modules and their homotopical applications are the focus of Part I of this book.

1.4 Filtered spaces

Once the 2-dimensional theory had been developed it was easy to conjecture, particularly considering work of J.H.C. Whitehead in [Whi49], that the theory in all dimensions should involve filtered spaces. An approach to algebraic topology via filtered spaces is unusual, so it is worth explaining here what is a filtered space and how this notion fits into algebraic topology.

A *filtered space* X_* is simply a topological space X and a sequence of subspaces:

$$X_* :\ X_0 \subseteq X_1 \subseteq X_2 \subseteq X_1 \cdots \subseteq X_n \subseteq \cdots \subseteq X_\infty = X.$$

A standard example is the filtration of a geometric simplicial complex by its skeleta: X_n is the union of all the simplices in X of dimension $\leqslant n$. More generally, X would be a CW-complex, the generalisation of the finite cell complexes in [Bro06], and X_n is the union of all the cells of dimension $\leqslant n$. Here X_{n+1} is obtained from X_n by attaching cells of dimension $n+1$.

There are other simple examples, which are important for us. One is when (X, A, x) is a pointed pair of spaces, i.e. $x \in A \subseteq X$, and $n \geqslant 2$. Then we have a filtered space $X_*^{[n]}$ in which $X_i^{[n]}$ is $\{x\}$ for $i = 0$, A for $0 < i < n$ and is X for $i \geqslant n$. It may be asked: why go to this bother? Why not just stick to the pair (X, A, x)? The answer is that for $n \geqslant 3$ we want to use conditions such as $\pi_i(X, A, x) = 0,\ 1 < i < n$, and to this end we in some sense 'climb up' the above filtration $X_*^{[n]}$.

Another geometric example of filtered space is when X is a smooth manifold and $f : X \to \mathbb{R}$ is a smooth map. Morse theory shows that f may be deformed into a map g which induces what is called a handlebody decomposition of X, which is a filtration of X in which X_{n+1} is obtained from X_n by attaching 'handles' of type $n+1$. This area is explored by methods related to ours in [Sha93, Chapter VI]. A further refinement is the notion of *stratification* of a space, which occurs in singularity theory.

It is of course standard to consider the simplicial singular complex SX of a topological space X, to obtain invariants from this, and then if X has a filtration to make further developments to get information on the filtered invariants. An example of this kind is when X is a CW-complex and we use the skeletal filtration.

1.5 Crossed complexes

Central to our work is the association to any filtered space X_* of its *fundamental crossed complex* ΠX_*. This is defined using the fundamental groupoid $\pi_1(X_1, X_0)$ and the family of relative homotopy groups $\pi_n(X_n, X_{n-1}, x)$ for all $x \in X_0$ and $n \geqslant 2$, and generalises the crossed module of a pointed pair of spaces.

A *crossed complex* C over C_1, where C_1 is a groupoid with object set C_0, is a sequence

$$\cdots \longrightarrow C_n \xrightarrow{\delta_n} C_{n-1} \xrightarrow{\delta_{n-1}} \cdots\cdots \xrightarrow{\delta_3} C_2 \xrightarrow{\delta_2} C_1$$

of morphisms of groupoids over C_0 such that for $n \geqslant 2$ C_n is just a family of groups, abelian if $n \geqslant 3$; C_1 operates on C_n for $n \geqslant 2$; $\delta_{n-1}\delta_n = 0$ for $n \geqslant 3$; and other axioms hold which we give in full in Chapter 7. The axioms are in fact those universally satisfied by ΠX_*, as we prove in Chapter 13.

One crucial point is that $\delta_2 : C_2 \to C_1$ is a crossed module (over the groupoid C_1). The whole structure has analogies to a chain complex with a groupoid of operators; this analogy is worked out in terms of a pair of adjoint functors in Sections of Chapter 7. However in passing from a crossed complex to its associated chain complex with operators some structure is lost. Crossed complexes have better realisation properties than these chain complexes, as the crossed module part in dimensions 1 and 2 in crossed complexes allows the modelling of homotopy 2-types, unlike the chain complexes.

In the case X_0 is a singleton, which we call the *reduced* case, the construction of ΠX_* is longstanding, but the general case was defined by Brown and Higgins in [BH81b, BH81a].

1.6 Why crossed complexes?

• They generalise groupoids and crossed modules to all dimensions, and the functor Π is classical, involving relative homotopy groups.

• They are good for modelling CW-complexes.

• Free crossed resolutions enable calculations with small CW-models of $K(G,1)$s *and their maps* (Whitehead, Wall, Baues), and special kinds of crossed complexes yield useful representations of cohomology of a group, [ML79, Hue80, Lue81].

• Crossed complexes give a kind of 'linear model' of homotopy types which includes all 2-types. Thus although they are not the most general model by any means (they do no contain quadratic information such as Whitehead products), this simplicity makes them easier to handle and to relate to classical tools. The new methods and results obtained for crossed complexes can be used as a model for more complicated situations. This is how a general n-adic Hurewicz Theorem was found [BL87a].

• They are convenient for some *calculations* generalising methods of computational group theory, e.g. trees in Cayley graphs. We explain some results of this kind in Chapter 11.

• They are close to the traditional chain complexes with a group(oid) of operators, as shown in MD6), and are related to some classical homological algebra (e.g. *identities among relations for groups*). Further, if SX is the simplicial singular complex of a space, with its skeletal filtration, then the crossed complex $\Pi(SX)$ can be considered as a slightly non commutative version of the singular chains of a space. However crossed complexes have better realisation properties than the related chain complexes.

• The category of crossed complexes has a monoidal structure suggestive of further developments (e.g. *crossed differential algebras*).

• They have a good homotopy theory, with a *cylinder object, and homotopy colimits*. There are homotopy classification results in Chapter 12 generalising a classical theorem of Eilenberg-Mac Lane.

• They have an interesting relation with the Moore complex of simplicial groups and of simplicial groupoids, [Ash88, NT89, EP97].

1.7 Higher Homotopy van Kampen Theorem

The reason why we deal with filtered spaces is the following. It is well known that many useful and geometrically interesting topological spaces are built by processes of gluing, or what we call colimits, from simpler spaces. Very often these simpler spaces have a natural, perhaps simple, filtration so that we often get an induced filtration on the colimit. One of our central results is a Higher Homotopy van Kampen Theorem (HHvKT), which involves the fundamental crossed complex functor Π of previous subsections. The theorem shows that for a filtered space built as a 'nice' colimit of so called *connected* filtered spaces, not only is the colimit also connected but we can compute the homotopical invariant Π of the colimit as a colimit of the Π of the individual pieces from which the colimit is built, and the morphisms between them.

From this result we deduce, for example:

(i) the Brouwer degree theorem (the n-sphere S^n is $(n-1)$-connected and the homotopy classes of maps of S^n to itself are classified by an integer called the *degree* of the map);

(ii) the relative Hurewicz theorem, which usually relates relative homotopy and homology groups, but is seen here as describing the morphism $\pi_n(X, A, x) \to \pi_n(X \cup CA, x)$ when (X, A) is $(n-1)$-connected;

(iii) Whitehead's theorem (1949) that $\pi_2(X \cup \{e^2_\lambda\}, X, x)$ is a free crossed $\pi_1(X, x)$-module;

(iv) a generalisation of that theorem to describe the crossed module $\pi_2(X \cup_f CA, X, x) \to \pi_1(X, x)$ as induced by the morphism $f_* : \pi_1(A, a) \to \pi_1(X, x)$ from the identity crossed module $\pi_1(A, a) \to \pi_1(A, a)$; and

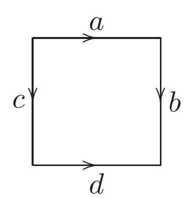
(v) a coproduct description of the crossed module $\pi_2(K \cup L, M, x) \to \pi_1(M, x)$ when $M = K \cap L$ is connected and $(K, M), (L, M)$ are 1-connected and cofibred.

We explain later other applications of crossed complexes in algebraic topology. However we are unable to prove many of these results in the sole context of crossed complexes, and have to venture into new structures on *cubical sets*. The next subsection begins the explanation of that background.

1.8 Cubical sets with connections

An extra structure which we needed for $\rho_2(X, A, X_0)$ in order to express the notion of cube with commutative boundary was what Chris Spencer and I called *connections*, because of a relation with path-connections in differential geometry. The background is as follows.

Even in ordinary category theory we need the 2-dimensional notion of commutative square:



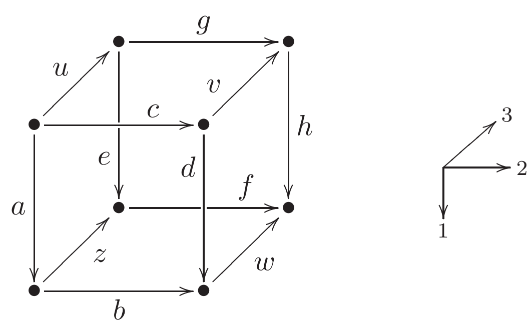
$ab = cd \quad (a = cdb^{-1}$ in the groupoid case$)$

An easy result is that any composition of commutative squares is commutative. For example, in ordinary equations:

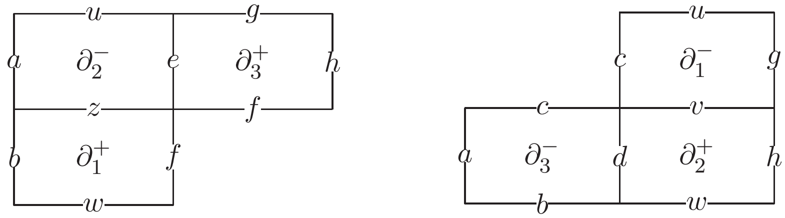
$$ab = cd, ef = bg \text{ implies } aef = abg = cdg.$$

The commutative squares in a category form a *double category*, and this fits with Diagram (multcomp).

What is a commutative cube, or, more precisely, what is a cube with commutative boundary? Here is a diagram of a 3-cube with labelled and directed edges:



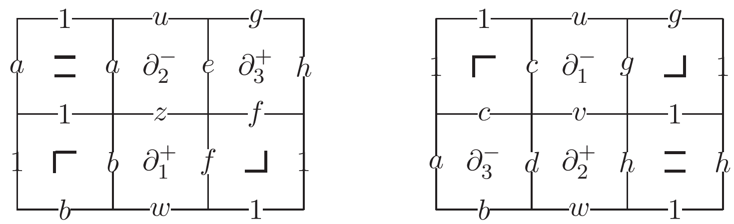
A prospective 'commutativity formula' involving just the edges is easy to write down. However, we want a 2-dimensional notion of the 'commutativity of the faces'. The problem is that a cube has 6 faces divided into odd and even ones, which fit together as shown in Diagram (*):



(*)

even faces odd faces

Unfortunately, the possible 'compositions of the even and of the odd faces' as above do not make sense, and the edges along the two 'boundaries' do not agree. We need canonical ways of filling in the corners, which correspond to the idea that in 2-dimensional algebra we need the extra possibility of turning left or right, as well as of carrying straight on. So we add to the available structure some special squares called *thin* squares which complete the diagram (*) as shown:



(**)

Here the squares $\equiv$ are identity elements for the horizontal composition $+_2$ but the squares $\ulcorner$, $\lrcorner$, which are called *connections*, have to be explained in detail. We say that the original cube is *commutative*, or, more precisely, has commutative shell, if the completed composite elements in diagram (**) are equal.

We also need sufficient axioms to be able to prove that any well defined composition of commutative cubes is commutative. We give these axioms for this dimension in Chapter 6. The idea has then to be carried through in all dimensions. This is part of the work of Chapter 13, and clearly needs new ideas to avoid what might seem impossible complications. The use of cubical sets with connections and compositions is again a departure from tradition.

1.9 Why cubical homotopy omega-groupoids with connections?

Standard algebraic topology uses a singular complex SX of a topological space, develops homology, and then if X has a filtration, needs to work to relate the algebraic topology to the filtered structure. Our approach is to take a singular complex which depends on the filtration; it is also necessary to work cubically[3].

It was easy to conjecture that to generalise the construction $\rho_2(X, A, X_0)$ given above, we should consider a filtered space X_* and the family $R_n X_*$ of sets of maps $I^n \to X$ which map the r-skeleton of I^n into X_r, i.e. the filtered maps $I^n_* \to X_*$; and then take homotopy classes of such relative to the vertices of I^n, giving a quotient map $p : RX_* \to \rho X_*$. Both RX_* and ρX_* have easily the structure of cubical set in terms of well known face and degeneracy maps. Cubical theory was initiated by D.M. Kan in 1955, but was abandoned for the simplicial theory, on which there is now an enormous literature. Nonetheless, multiple compositions are difficult simplicially, while the natural context for them is cubical. Such a cubical approach does move away from standard algebraic topology. Also it was necessary to introduce into the cubical theory the notion of connections in all dimensions.

It was not found easy to prove a central feature of our work that the easily defined multiple compositions in RX_* were inherited by ρX_*. A further difficulty was to relate the structure held by ρX_* to the crossed complex ΠX_* traditional in algebraic topology. These proofs needed new ideas and are stated and proved in Chapter 14.

Here are the basic elements of the construction.

I^n_*: the n-cube with its skeletal filtration.

Set $R_n X_* = \mathsf{FTop}(I^n_*, X_*)$. This is a *cubical set with compositions, connections, and inversions*.

For $i = 1, \dots, n$ there are standard:

face maps $\partial^\pm_i : R_n X_* \to R_{n-1} X_*$;

degeneracy maps $\varepsilon_i : R_{n-1} X_* \to R_n X_*$

connections $\Gamma^\pm_i : R_{n-1} X_* \to R_n X_*$

compositions $a \circ_i b$ defined for $a, b \in R_n X_*$ such that $\partial^+_i a = \partial^-_i b$

inversions $-_i : R_n \to R_n$.

The connections are induced by $\gamma^\alpha_i : I^n \to I^{n-1}$ defined using the monoid structures $\max, \min : I^2 \to I$. They are essential for many reasons, e.g. to discuss the notion of *commutative cube*.

These operations have certain algebraic properties which are easily derived from the geometry and which we do not itemise here – see for example [AABS02]. These were listed first in the Bangor thesis of Al-Agl [AA89]. (In the paper [BH81b] the only basic connections needed are the Γ^+_i, from which the Γ^-_i are derived using the inverses of the groupoid structures.)

Here we explain why we need to introduce such new structures.

• The functor ρ gives a form of *higher homotopy groupoid*, thus confirming the visions of topologists of the early 20th century of higher dimensional nonabelian forms of the fundamental group.

• They are equivalent to crossed complexes, and this equivalence is a kind of cubical and nonabelian form of the Dold-Kan theorem, relating chain complexes with simplicial abelian groups.

• They have a clear *monoidal closed structure*, and notion of homotopy, from which one can deduce analogous structures on crossed complexes, with detailed formulae, using the equivalence of categories.

• It is easy to relate the functor ρ to tensor products, but quite difficult to do this for Π.

• Cubical methods, unlike globular or simplicial methods, allow for a simple *algebraic inverse to subdivision*, involving multiple compositions in many directions which are crucial for the proof of our HHvKT in Chapter 14.

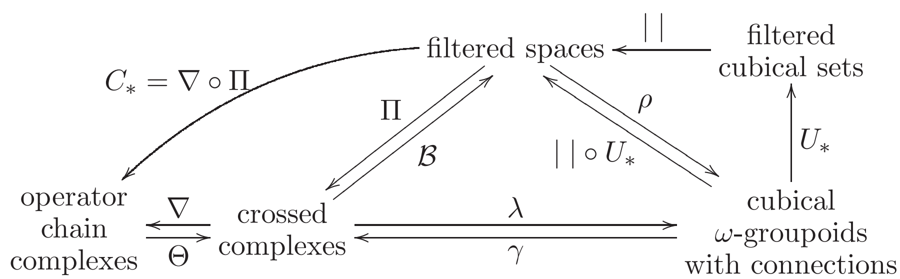
• The additional structure of 'connections', and the equivalence with crossed complexes, allows the notion of *thin cube* which subsumes the idea of commutative cube, and yields the proof that *multiple compositions of thin cubes are thin*. This last fact is another key component of the proof of the HHvKT.

• The cubical theory gives a construction of a *(cubical) classifying space* $BC = (\mathcal{B}C)_\infty$ of a crossed complex C, which generalises (cubical) versions of Eilenberg-Mac Lane spaces, including the local coefficient case.

• There is a current *resurgence of the use of cubes* in for example combinatorics, algebraic topology, and concurrency, see for example [GNAPGP88, BJT09, BP02, Mal09, Gou03].

1.10 Diagram of the relations between the main structures

The complete and intricate story has its main facts summarised in the following diagram and comments:



(Main Diagram)

in which

MD 1) the categories FTop of filtered spaces, Crs of crossed complexes and ω-Gpd of ω-groupoids, are monoidal closed, and have a notion of homotopy using $\otimes$ and unit interval objects;

MD 2) ρ, Π are homotopical functors (that is they are defined in terms of homotopy classes of certain maps), and preserve homotopies;

MD 3) λ, γ are inverse adjoint equivalences of monoidal closed categories, and λ is a kind of 'nerve' functor;

MD 4) there is a natural equivalence $\gamma\rho \simeq \Pi$, so that either ρ or Π can be used as appropriate;

MD 5) ρ preserves certain colimits and certain tensor products, and hence so also does Π;

MD 6) the category Chn of chain complexes with a groupoid of operators is monoidal closed, and ∇ is a monoidal functor which has a right adjoint Θ;

MD 7) by definition, the *cubical filtered classifying space* is $\mathcal{B} = | \, | \circ U_* \circ \lambda$ where U_* is the forgetful functor to filtered cubical sets using the filtration of an ω-groupoid by skeleta, and $| \, |$ is geometric realisation of a cubical set;

MD 8) there is a natural equivalence $\Pi \circ \mathcal{B} \simeq 1$;

MD 9) if C is a crossed complex and its cubical classifying space is defined as $BC = (\mathcal{B}C)_\infty$, then for a CW-complex X, and using homotopy as in MD1) for crossed complexes, there is a natural bijection of sets of homotopy classes

$$[X, BC] \cong [\Pi X_*, C] \tag{MD9}$$

2 Structure of the book

Because of the complications set out above in the Main Diagram, and in order to communicate the basic intuitions, we divide our account into three parts, each with an Introduction.

In Part I we give some history of work on the fundamental group and groupoid, in particular explaining how the van Kampen theorem with a set of base points gives a method of computation of fundamental groups. It was the extension of this classical theorem from groups to groupoids that led to the question of the putative uses of groupoids in higher homotopy theory, and so to the homotopy double groupoid of pair of spaces (X, A) with a set X_0 of base points, a key tool for proving the 2-dimensional van Kampen theorem in Chapter 6.

However Chapters 2-5 are concerned entirely with the single base point case, in order to establish the main features of the algebra and applications of crossed modules of groups, as extensions to dimension 2 of well known work on the fundamental group.

The remarkable fact is that we can calculate with these 2-dimensional structures and apply these calculations to topology using a 2-dimensional version of the van Kampen theorem for the fundamental group.

We give a substantial account of this 2-dimensional theory because the step from dimension 1 to dimension 2 involves a number of new ideas for which the reader's intuition needs to be developed. In particular, calculation with crossed modules requires some extensions of combinatorial group theory, for example to induced crossed modules. Finally in this Part, the proof of the van Kampen type theorem for crossed modules, involves a notion of *homotopy double groupoid*, based on composing squares with common edges. The intuition for this construction was the start of the theory of this book.

One aim is to give references to work on crossed module which are relevant to our main theme. However work on crossed modules in a variety of areas has been burgeoning over the last ten years, as a web search shows, and so we cannot hope to give here a full survey of their uses.

The aim of Part II is to give a kind of handbook of applications of *crossed complexes*, assuming some major properties which are proved in Part III. The theory extends many basic results in homotopy theory, such as the relative Hurewicz theorem. Among results found in no other text on algebraic topology is the homotopy classification theorem referred to above in MD 9).

In Part III we define the *cubical ω-groupoids with connections* whose properties are the power behind the applications of crossed complexes. In principle, and this would be the logical order, Part III can be read independently of the previous parts, referring back for some basic definitions.

In Part IV we give a 'Conclusion', in which we try to evaluate what has been done, to point the reader to some other current directions, and to indicate some of the many things yet to be investigated with these tools.

Appendices give some categorical results which may not be so easy to access in the form needed.

Prerequisites and Reading plan

The aim is for large parts of this book to be readable by a graduate student acquainted with general topology, the fundamental group, notions of homotopy, and some basic methods of category theory. Many of these areas, including the concept of groupoid and its uses, are covered in Brown's text 'Topology and Groupoids', [Bro06]. The only theory we have to assume for the homotopy classification theorem in Chapter 12 and in some applications in Chapter 14 is some results on the geometric realisation of cubical sets.

Some aspects of category theory perhaps less familiar to a graduate student are summarised in an Appendix, particularly the notion of representable functor, the notion of dense subcategory, and the preservation of colimits by a left adjoint functor. This last fact is a basic tool of algebraic computation for those algebraic structures which are built up in several levels, since it can often show that a colimit of such a structure can be built up level by level.

We make no use of classical tools such as simplicial approximation, but some knowledge of homology and homotopy of chain complexes could be useful at a few points, to help motivate some definitions.

We feel it is important for readers to understand how this theory derives from the basic intuitions and history of algebraic topology, and so we start Part I with some history. After that, historical comments are given in Notes at the end of each chapter.

This book is designed to cater for a variety of readers.

Those with some familiarity with traditional accounts of relative homotopy theory could skip through the first two chapters, and then turn to Chapter 6, and its key account of the homotopy double groupoid $\rho(X, A, C)$ of a pair of spaces (X, A) with a set C of base points. The fact that this construction avoids both the choices that are usually made in defining second relative homotopy groups, and also the composition on a single line, is the key to the use of this construction in proving a 2-dimensional van Kampen Theorem, and so giving the applications in Chapters 4 and 5. Part I does develop a lot of the algebra and applications of crossed modules (particularly coproducts and induced crossed modules) and the full story of these can be skipped over.

Part II gives the major applications of crossed complexes, with the proofs of key results given in part III, using the techniques of cubical ω-groupoids.

Finally, those who want the pure logical order could read the book starting with Part III, and referring back for basic definitions where necessary.

Notes

[1]p. 1 The naming of theorems is a difficult matter, and subject to disagreement. Here we give some history. The first result describing the fundamental group of a union was that of Seifert in [Sei31], for the union of two connected subcomplexes with connected intersection of a simplicial complex. The next result was that of van Kampen in [Kam33]. He also gives a formula for the case of nonconnected intersection. His proofs are difficult to follow.

The start of the modern approach is the paper of Crowell [Cro59], based on lectures of R.H. Fox, which used the term colimit and the proof was by verification of the universal property.

Olum in [Olu58] gave a proof for the case of a union of two sets with connected intersection using nonabelian cohomology with coefficients in a group, and he also carefully analyses van Kampen's local conditions. The Mayer-Vietoris type sequence given by Olum was extended in [Bro65], so that the fundamental group of the circle could be computed. It was then found that a more powerful result with simpler proof could be obtained using groupoids, [Bro67], giving the fundamental groupoid on a set of base points for the case of nonconnected intersection of two open sets. This was suggested by the use by Higgins in [Hig64] of free product with amalgamation of groupoids. Thus an aim to compute a fundamental group was reached by first computing a larger structure, a fundamental groupoid on a set of base points, and then giving methods of a combinatorial character for computing the group from the larger structure. A generalisation to unions of families was given in [BRS84]. A combination of the method of Olum with the use of groupoids is given in [BHK83].

A general result for the non connected case but still only for groups is in [Wei61], using the notion of the nerve of the cover to describe graph theoretic properties of the components of the intersections of the open sets.

All these insights have been important for the generalisations to higher dimensions. Thus we find it convenient to refer to theorems of these types as *van Kampen theorems*.

We say more later on other extensions and analogues of the 1-dimensional theorem.

[2]p. 3 The reader should be warned that the account of crossed modules in the second edition of [ML71] omits the rule CM2) and its relation with the interchange law for group objects in groupoids.

[3]p. 8 This work progressed in the 1970s when we abandoned the attempt to define a 'higher homotopy groupoid' for a space and instead worked with pairs of spaces and for higher dimensions with filtered spaces. This enabled us to construct the cubical homotopy ω-groupoid $\rho(X_*)$ which is at the heart of this work. Nowadays this would be called a 'strict' ω-groupoid. There is a tendency to call the simplicial complex SX the 'fundamental ∞-groupoid' of the space X, and even to label it ΠX, see for example [Lur09]. Our notation ΠX_* is intended to reflect the close relation to traditional concepts in homotopy theory, the relative homotopy groups. In a similar manner, the notation $\Pi\mathbf{X}$ is used in [BL87b] to denote the strict structure of what is there termed the fundamental cat^n-group of an n-cube of spaces $\mathbf{X}$.

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