Algebraic Topology and the Nonabelian Tensor Product of Groups

The non abelian tensor product of groups was found and seen to be important for its connection with the Blakers-Massey Theorem in Algebraic Topology.

Theorem (Blakers-Massey 1953)*Let* $X = A \cup B$ *be the union of path connected open sets with path connected intersection* C*. Suppose that the pairs* $(A, C), (B, C)$ *are respectively* $p-1, q-1$ *connected. Then the triad* $(X; A, B)$ *is* $p+q-2$ *connected, and if* $p, q \geq 3$ *then the triad homotopy group* $\pi_{p+q-1}(X; A, B)$ *is isomorphic to the tensor product* $\pi_p(A, C) \otimes \pi_q(B, C)$*, and the isomorphism is given by the generalized Whitehead product.*

However this gives no answer for the case p or q is 2. The answer for this case is given for the case $p, q = 2$ by Brown-Loday "Excision homotopique en basse dimension", *C.R. Acad. Sci. Paris Sér.* I 298 (1984) 353-356, simply by replacing the usual tensor product by the non abelian tensor product of the $\pi_1(C)$-modules $\pi_p(A, C), \pi_q(B, C)$ (crossed if p or q is 2). For this the account given in our Introduction has to be generalised to the case of groups G, H acting on each other (on the left) and this is the case if G, H are P-crossed modules acting on each other via P. The case p or q is 2 is given in essence in Brown-Loday "Homotopical excision, and Hurewicz theorems, for n-cubes of spaces", *Proc. London Math. Soc.* (3) 54 (1987) 176-192, which relies on Brown-Loday, "Van Kampen theorems for diagrams of spaces", *Topology* 26 (1987) 311-334. See also Brown, R., Triadic Van Kampen theorems and Hurewicz theorems, *Contemporary Mathematics*, Vol. 96, 1989, 39-57.

The triadic Hurewicz theorem says that under the above connectivity conditions, but without assuming $X = A \cup B$, the triad homology group $H_{p+q-1}(X; A, B)$ is obtained from $\pi_{p+q-1}(X; A, B)$ by factoring out the operations and the generalized Whitehead product.

See also the last section of

R. Brown and R. Sivera, 'Algebraic colimit calculations in homotopy theory using fibred and cofibred categories', Theory and Applications of Categories, 22 (2009) 222-251.