SOME PROBLEMS IN NON-ABELIAN HOMOTOPICAL AND HOMOLOGICAL ALGEBRA

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Part of the motivation of the paper was to give a kind of survey of the area in terms of what had not been done, thus allowing a lighter touch on what had been done.
A number of the problems came from grant proposals unsupported as either 'irrelevant to mainstream algebraic topology', or because 'calculating integral homotopy types is not a central problem of homotopy theory'. It seemed worthwhile making use of the work involved in the grant proposals by collecting these problems together for the 1990 Conference Proceedings.

It now (September, 2001) seems useful to make these problems generally available on the xArchiv in view of a recent surge of interest in the wide applications of higher categorical structures. Some references to recent papers and problems solved have been added at this date (September, 2001).
Further discussion and details of papers published later than 1989 are also given in the web article
Higher dimensional group theory http://www.bangor.ac.uk/ mas010/hdaweb2.htm
and the accompanying article on the non abelian tensor product.

Introduction

The aim of this paper is to convey some impression of the extent of an area of non-Abelian homotopical and homological algebra, by giving some of the problems, of varying degrees of difficulty and of precision, which I have come across over the years and in which progress would be desirable.

In his address to the International Congress of Mathematicians at Harvard in 1950, J.H.C. Whitehead said (Whitehead,1950b):

> The ultimate object of algebraic homotopy is to construct a purely algebraic theory which is equivalent to homotopy theory in the same way that 'analytic' is equivalent to 'pure' projective geometry.

Whitehead achieved a purely algebraic description of the homotopy type of any 3-dimensional complex, and of any simply connected 4-dimensional complex (Whitehead, 1950a). Mac Lane and Whitehead (1950) also gave a manageable description of 2-types in terms of crossed modules. Crossed modules are a central feature of recent extensions of the Van Kampen Theorem and of applications of the related algebra to non-Abelian homological algebra. They are also used in considerable recent work of Baues on the homotopy classification of spaces and of maps. There is now a variety of algebraic theories which can model aspects of homotopy theory, so that it can be argued that we are much nearer to achieving Whitehead's aim. We shall give problems associated with two generalisations of crossed modules: crossed complexes, and cat^n-groups.

1 Problems on crossed complexes

Crossed complexes are a kind of model of chain complexes with operators, but including the non-Abelian features of crossed modules in dimensions 1 and 2. The canonical example is the fundamental crossed complex πX_* of a filtered space

$$X_* : X_0 \subseteq X_1 \subseteq \cdots \subseteq X_n \subseteq \cdots \subseteq X_\infty = X. \tag{1}$$

This crossed complex πX_* consists of the fundamental groupoid $\pi_1 X_* = \pi_1(X_1, X_0)$ (i.e. the path groupoid with X_0 as set of base points) together with for each $n \geqslant 2$ the family $\pi_n X_*$ of relative homotopy groups $\pi_n(X_n, X_n{-}_1, x)$ for all $x \in X_0$, and the usual boundary maps $\pi_n X_* \to \pi_n{-}_1 X_*$ and action of $\pi_1 X_*$ on $\pi_n X_*$. The axioms universally satisfied by this example are the axioms for a crossed complex. In particular, the part $\pi_2 X_* \to \pi_1 X_*$ has the structure of crossed module (over a groupoid).

Morphisms of crossed complexes are easily defined, and so we have a category Crs of crossed complexes. The fundamental crossed complex then gives a functor π : (filtered spaces) $\to$

Crs, which encapsulates standard features of relative homotopy groups and change of base point.

It is shown in Brown-Higgins (1987) that the category Crs admits a symmetric monoidal closed structure with internal hom functor $CRS(-,-)$ and tensor product $-\otimes-$, so that there is an exponential law

$$\mathsf{Crs}(A\otimes B, C)\cong \mathsf{Crs}(A, CRS(B,C)). \tag{2}$$

The internal hom $CRS(B,C)$ consists of the morphisms $B\to C$ in dimension 0, the homotopies of morphisms in dimension 1, and the higher homotopies in higher dimensions.

The tensor product of crossed complexes is closely related to products of filtered spaces. It is shown in Brown-Higgins (1989b) that there is a natural transformation

$$\eta : \pi X_* \otimes \pi Y_* \to \pi(X_*\otimes Y_*) \tag{3}$$

which is the identity in dimension 0 , and where the tensor product of filtered spaces is defined as should be expected as the filtration of $X_\infty \times Y_\infty$ which in level n is

$$\bigcup_{p+q=n} X_p \times Y_q. \tag{4}$$

Further the natural transformation η is an isomorphism if X_* and Y_* are the skeletal filtrations of CW-complexes. In order to see the complications of the tensor product that this implies, the reader should work out for example the crossed complex $\pi(\mathbf{E}^2\otimes\mathbf{E}^2)$ where E^2 is the 2-disc with cell structure $e^0\cup e^1\cup e^2$. In fact the crossed complex $A\otimes B$ is generated by elements $a\otimes b, a\in A_m, b\in B_n$, with the relations determined by those satisfied in the universal examples $\pi(\mathbf{E}^m\otimes\mathbf{E}^n), m,n\geqslant 0$ (as well, of course, as the laws for a crossed complex).

All this is analogous to the classical Eilenberg-Zilber Theorem on the normalized chains of the product of simplicial sets. An easy application of acyclic models shows that if K and L are simplicial sets then there is a natural homotopy equivalence of crossed complexes

$$\chi : \pi|K|\otimes\pi|L| \to \pi|K\times L|, \tag{5}$$

where the geometric realisations are given their skeletal filtrations as CW-complexes. The following is then a problem purely of calculation.

Problem 1.1. *Give explicit formulae for χ; for a homotopy inverse to χ; and for the natural homotopies involved, analogous to those well known for the case of chain complexes.*

[This problem has been completely solved in [27]. Applications are given in [9].]

The detailed properties of the maps and homotopies of the ordinary Eilenberg-Zilber theorem have led to a result in Brown, 1967, which is now (for example in Lambe-Stasheff,

1987) called the Perturbation Lemma. It is applied in these and other papers to computations of twisted tensor products, and is used elsewhere (e.g. in Kassell, 1988) for other calculations with chain complexes.

Problem 1.2. *Is there a non-Abelian perturbation lemma for constructing non-Abelian twisted tensor products from fibrations?*

I am not convinced that this can be done, but it is interesting to determine the extent to which the Abelian techniques can be carried over to the non-Abelian case, in such a way as to prove useful.

The tensor product of crossed complexes is symmetric, as stated above (Brown-Higgins, 1987). So if K is a simplicial set, then we can consider the non-commutativity of the diagonal map $\Delta : \pi|K| \to \pi|K| \otimes |K|$. If T is the twisting map $A \otimes B \to B \otimes A$, then there is a natural homotopy $T\Delta \simeq \Delta$, by the usual acyclic models argument. This look like the beginnings of a theory of non-Abelian Steenrod cohomology operations.

Problem 1.3. *Does such a theory exist and does it hold any surprises?*

By contrast, Baues (1989b) gives an obstruction to the existence of a Pontrjagin square with local coefficients.

One use of chain complexes is in defining Kolmogorov-Steenrod homology. One takes the usual net of polyhedra defined as the nerves of open covers of a space X, with maps between them induced by choices of refinements. The result is a homotopy coherent diagram of polyhedra. It is shown by Cordier, 1987, that a strong homology theory results by taking the chain complexes of this net, and forming the chain complex which is the homotopy inverse limit.

Problem 1.4. *What sort of strong homology theory results from using the fundamental crossed complexes of the nerves instead of the chain complexes?*

The homotopy inverse limit of these crossed complexes exists from general considerations (see Cordier-Porter, 1986), since crossed complexes in fact form also a simplicially enriched category. This may be proved by using the monoidal closed structure on Crs and the equivalence between Crs and the category of simplicial T-complexes (see below). This particular homotopy inverse limit has of course a fundamental groupoid, and various homology groups. The former seems to be a kind of "strong fundamental groupoid", and the latter seem to be a kind of strong "homology of the universal cover", which could be appropriate to cases where the universal cover itself does not in fact exist. (We have used the term *Kolmogorov-Steenrod homology* since, as pointed out to me by G.Chogoshvili (private communication), there are papers of Kolmogorov in this area prior to those of Steenrod.)

There are some nice results for the fundamental group and the fundamental groupoid of orbit spaces (Armstrong, 1982; Higgins-Taylor, 1982; Taylor, 1988; Brown-Higgins, 1986; Brown, 1988b). Higgins-Taylor give an expression for the fundamental crossed complex of an orbit space in the cellular case, deriving it from the freeness assumptions which arise in this case.

Problem 1.5. *Find results on the fundamental crossed complex of an orbit space of a filtered space analogous to those for the fundamental groupoid of an orbit space given in Brown, 1988b, section 9.8.*

The aim here is of course to find higher homotopy information on orbit spaces.

It is not apparent from what we have said so far, but our crossed complex approach to non-Abelian homotopical algebra was initially motivated by considering known results for the fundamental group or groupoid, and searching for higher dimensional versions, using some notion of "higher dimensional groupoid". One aim of using groupoids here was to get round the old and standard argument that "higher dimensional groups" are just Abelian groups. Thus the aim was to produce "higher homotopy groupoids" which were expected to satisfy higher dimensional Van Kampen Theorems of a kind which would lead to some explicit computations. For an intuitive account of the background to these ideas, see Brown, 1982, 1988a. It may have been a general reluctance to make this extension from groups to groupoids that led to models of the kind proposed not having been considered much earlier. The case for groupoids can now be seen to be strong (Brown, 1987).

The construction of the natural transformation $\eta : \pi X_* \otimes \pi Y_* \to \pi(X_* \otimes Y_*)$ of (3) derives from a corresponding and simply constructed natural transformation $\eta' : \rho X_* \otimes \rho Y_* \to \rho(X_* \otimes Y_*)$ for the fundamental ω-groupoid ρX_* of a filtered space. A construction of ρX_* is given in Brown-Higgins, 1981b, but, as pointed out in Brown-Higgins, 1989b, it is convenient to change conventions slightly and take the elements of $\rho_n X_*$ to be filter homotopy classes *rel vertices* of filtered maps $I^n_* \to X_*$. The transformation η' is given by $[f] \otimes [g] \mapsto [f \otimes g]$.

The category of ω-groupoids is equivalent to the category of crossed complexes (Brown-Higgins, 1981a). This equivalence may be regarded as an algebraic version of the equivalence between various ways of defining the relative homotopy groups. A further point is that the constructions on ω-groupoids are more geometrical and canonical than those on crossed complexes. The latter constructions tend to involve choices in the same way as does the definition of relative homotopy groups $\pi_n(X, A, x)$ by maps of cubes I^n, namely choices: of base point on I^n; of which faces of I^n are to be mapped to the base point x of X; and of which faces are mapped into A. We mention that it is not clear that the conventions for base points and ordering chosen in Brown-Higgins, 1981a, are the optimal ones, from the point of view of the shape of the final formulae for say the homotopy addition lemma and the tensor product, and a minor problem is to rewrite the specific formulae which involve these results after making some decision on optimal conventions.

The category of ω-groupoids is one of a family of categories equivalent to crossed complexes, the others being cubical T-complexes (Brown-Higgins, 1981c), ∞-groupoids (Brown-Higgins, 1981d), simplicial T-complexes (Dakin, 1976, and Ashley, 1988), polyhedral T-

complexes (Jones, 1988), and simplicial T-groupoids whose complex of objects is constant (Nan Tie, 1989a). The advantages of a cubical rather than simplicial approach are that cubical sets have easily defined multiple composition structures, which are difficult to handle in the simplicial case, and that cubical sets behave well for homotopies and higher homotopies. The former fact is reflected in the proof of the Generalised Van Kampen Theorem in Brown-Higgins, 1981b. The latter fact is reflected in the way in which the monoidal closed structure of crossed complexes is derived in Brown-Higgins, 1987, from an easily constructed monoidal closed structure for ω-groupoids, and the equivalence between ω-groupoids and crossed complexes. A further point in favour of ω-groupoids is that the monoidal closed structure is canonical, whereas that on crossed complexes does involve the explicit form chosen for the equivalence with ω-groupoids.

As mentioned above, the simplicial version of crossed complexes, namely Keith Dakin's *simplicial T-complexes* (Dakin, 1977), was shown equivalent to crossed complexes in Ashley, 1988. (See also Nan Tie, 1989b.) It is this equivalence which also yields a simplicial enrichment for the category of crossed complexes. However, the complexities of this equivalence have made it difficult to solve the following problem.

Problem 1.6. *Give an explicit description of the closed monoidal structure on the category of simplicial T-complexes which results from the above mentioned equivalence with crossed complexes.*

There are a number of open problems with the notion of polyhedral set due to Jones, 1988.

Problem 1.7. *Give conditions on the models of a category of polyhedral sets for its homotopy category of Kan complexes to be equivalent to the homotopy category of CW-complexes.*

Inevitably, there is also the problem of tensor products and higher homotopies for the category of polyhedral T-complexes.

Problem 1.8. *Give an explicit description of the symmetric monoidal closed structure on an appropriate category of polyhedral T-complexes which derives from its equivalence with crossed complexes.*

In principle this ought to be not unreasonable, since the models of polyhedral sets have the property enjoyed by cubes but not enjoyed by simplices, that the product of two models is again a model.

One of the motivations of the polyhedral theory was to give a proper account of some kind of higher dimensional Cayley complex of a group presentation, and to give application to Van Kampen diagrams in group theory (see Johnson, 1980, for the usual theory). The point is that if you are considering groups with a relation of the form $x^5 = 1$, then it seems appropriate to have a geometric model which allows pentagons. If you allow pentagons, what else has been let in the door? The polyhedral sets of Jones, 1988, give an account of such possible models.

Problem 1.9. *Pursue the group theoretic applications of polyhedral sets and polyhedral T-complexes.*

Another reason for pursuing a notion of polyhedral sets is that simplices and cubes are not the only family of "standard" polyhedra which occur in algebraic topology and homotopy theory. There is also for example Stasheff's family of polyhedra which occur in connection with A_n-spaces (Stasheff, 1963, see also Baues, 1980). It would be interesting to have a more algebraically defined category equivalent to the category of polyhedral T-complexes which has models the Stasheff complexes and their faces.

The equivalence of crossed complexes with ω-groupoids, and the closed monoidal structure on crossed complexes, is used in Brown-Higgins, 1989a, to construct, and to prove the homotopy properties of, a *classifying space functor* $B : \mathsf{Crs} \to$ *(spaces)*. The main result is that if X is a CW-complex with skeletal filtration X_*, and C is a crossed complex, then there is a weak homotopy equivalence of spaces

$$\theta : B(CRS(\pi X_*, C)) \to TOP(X, BC). \tag{6}$$

The CW-complex $B\pi X_*$ has: vertex set X^0; fundamental groupoid $\pi_1(X, X)$; and for $n \geqslant 2$, $\pi_n(B\pi X_*, x)$ is isomorphic as $\pi_1(X, x)$-module to $H_n(\tilde{X}_x)$, the homology of the universal cover of X based at x. So (6) generalises well known results on maps to an Eilenberg-Mac Lane space, including the local coefficient case. Further, there is a map $X \to B\pi X_*$ whose induced map on homotopy groups is the Hurewicz map (Brown-Higgins, 1981b). These results show that crossed complexes carry over to the non-Abelian case, but with the inclusion of the fundamental group and its actions, some of the uses of chain complexes, simplicial abelian groups, and the free abelian topological group on a CW-complex. These latter constructions arise in the homotopy theory of mapping spaces and configuration spaces.

Problem 1.10. *Find applications of these non-Abelian constructions to configuration space theory and mapping space theory.*

The important point here is that if X is a simplicial set, then $\pi|X|$ is a crossed complex of free type, and so is a non-Abelian analogue of the free abelian group on X.

The existence of the monoidal closed structure on crossed complexes has led Brown-Gilbert, 1989, to a notion of *braided crossed module.* This is a crossed module $\mathcal{P}$ with a monoid structure $\mathcal{P} \otimes \mathcal{P} \to \mathcal{P}$, satisfying the condition that with this structure P_0 becomes a group. Of course P_0 is in general non-trivial since we are considering crossed modules over groupoids, and so are not working only in the reduced, i.e. one vertex, case. It is proved in *loc. cit.* that the category of braided crossed modules is equivalent to the category of simplicial groups whose Moore complex is of length 2. Thus braided crossed modules give another algebraic model of integral homotopy 3-types. A similar model has been considered by Joyal and Tierney (unpublished) in the simply connected case. [It is claimed (2001) that their model also covers the non simply connected case.] The word "braided" is used because of the connection with braided categories in Joyal-Street, 1986, see also Freyd-Yetter, 1986. Note also that simplicial groups whose Moore complexes are of length 2 are shown in Conduché, 1984, to be equivalent to what he calls crossed modules of length 2.

The tenor of Brown-Gilbert, 1989, is that the closed structure on the category of crossed modules over groupoids allows for the definition of an automorphism structure for crossed modules, a structure more elaborate than that of a group. Such a structure is in fact a braided crossed module, which is therefore an appropriate model for the notion of a higher order symmetry.

Problem 1.11. *Push this analogy further, and use such higher order structures in other situations where groups are currently used.*

The simplest mathematical expression of symmetry is that the bijections of a set form a group. Moving up one stage, the automorphisms of a group G form part of the crossed module $\xi : G \to AutG$, where ξ is the inner automorphism map. This is well known, but maybe has not been too much exploited in this form. The study of automorphisms of crossed modules was initiated by Whitehead, 1948, and continued in Lue, 1977, and Norrie, 1987a,b,c. As stated above, such automorphisms form part of the structure of braided crossed module, or, alternatively, of crossed square (Norrie, *loc. cit*).

An ideal scenario for this area would be that such higher order symmetries, i.e. structures describing the symmetries of objects already associated with symmetries, would be proved appropriate for some applications to the real world, for example to particle physics. Some theoretical physicists are already thinking in terms related to these, i.e. that string theory really needs not just groupoids but also multiple groupoids. One would certainly think that polyhedral T-complexes, say, have a rich enough geometrical and algebraic structure to model complicated forms of interactions.

Crossed modules are a special case of crossed complexes, and it is easy to use the structure of the tensor product of crossed complexes to define the notion of braided crossed complex.

Problem 1.12. *Find an equivalence between braided crossed complexes and a full subcategory of the category of simplicial groups.*

It is a pity that the equivalence between crossed complexes and simplicial T-complexes is one of the hardest of the various equivalences of categories mentioned above. The advantage of having a basis in simplicial theory for these structures is that simplicial techniques are well embedded in the literature, whereas cubical techniques have a kind of *samidzat* existence, where the basic properties have not been properly exposed. For example, the reference that is used in several papers by Brown-Higgins for the crucial equivalence between the homotopy categories of Kan cubical sets and of CW-complexes is Hintze, 1973, which has not been published elsewhere. Of course, there are several known difficulties with the cubical theory. For example, cubical groups need not be Kan complexes. However it seems appropriate to give cubical sets the additional structure of connections, as defined in Brown-Higgins, 1981a, which makes them more related to the simplicial theory and at the same time keeps the advantages of the cubical approach.

Problem 1.13. *Give a complete exposition of the cubical theory, including realisations, and cubical groups, with full attention to the structure of connections introduced in Brown-Higgins, 1981a.*

[Tonks [28] has shown that cubical groups with connection are Kan complexes.]

The original motivation for the development of ω-groupoids is that the fundamental ω-groupoid functor ρ : (filtered spaces) $\to$ (ω-groupoids) may be defined and that this higher homotopy groupoid is useful in proving the Generalised Van Kampen Theorem (GVKT) for the fundamental crossed complex of a filtered space (Brown-Higgins, 1981b). We recall that this GVKT is proved by basic subdivision arguments which do not require any sophisticated machinery from outside this area. In particular, a corollary is the Relative Hurewicz Theorem in the form of a relation between $\pi_n(X, A)$ and $\pi_n(X \cup CA)$, and with a proof not assuming any homology theory.

Problem 1.14. *Apply these algebraic methods to other geometric areas where subdivision arguments are crucial.*

An example which comes to mind here is the work of Tsuboi, 1985, on foliations, which uses complicated mixtures of cubical and simplicial arguments, and also uses complicated amalgamations of formulae for the chain boundary of a cube. By contrast, a key feature of the proof of the above GVKT is the machinery of amalgamating, or composing, thin elements in an ω-groupoid. This allows for the handling of multiple compositions of homotopy addition lemma formulae, without writing down any formulae!

The main point of the Generalised Van Kampen Theorems is to find new algebraic structures appropriate to at least a partial description of a big object (in this case, a homotopy type) in terms of the small pieces from which it is built. That is, one wants an algebraic model of homotopy types, rather than the standard Postnikov system, in order to apply algebraic constructions, such as limits and colimits, to model the geometric constructions of spaces. This is analogous to the necessity for using homology groups, rather than numerical invariants, for describing the homological properties of spaces obtained as inverse limits.

Building big objects out of pieces is a common procedure. So one expects new applications from the algebra which arises in this context. It is also interesting to analyse the new features of this kind of algebra, in order to look for analogous uses elsewhere in mathematics and science.

Crossed complexes have been known for a long time to be appropriate for considering the representation of elements of the cohomology of groups (Huebschmann, 1980; Holt, 1979; Mac Lane, 1979; Lue, 1971, 1981), in a way which generalises the description of H^n in terms of extensions. However, no clear description of cup product has been given in this context. One presumes that such a description would involve the tensor product of crossed complexes.

The theory of crossed complexes of groups is a special case of a theory of crossed complexes for varieties of algebras (Lue, loc. cit.). Some background to these ideas is explained in Brown, 1984a, in terms of non-Abelian chains of syzygies. Thus crossed complexes can be defined in certain varieties of algebras and are appropriate for the representation of cohomology classes (Lue, 1971, see also Porter, 1987a,b). This suggests that such crossed complexes should admit a useful tensor product.

Problem 1.15. *Construct such an appropriate tensor product for crossed complexes of algebras.*

There are several snags. One is that in the group case, the tensor product is part of a monoidal closed structure for crossed complexes over groupoids, and such a closed structure certainly does not exist in the reduced (= one vertex) case, anymore than it does for groups. Indeed, the fact that groupoids admit a cartesian closed structure, and groups do not, is one of the standard arguments in favour of groupoids over groups. One would therefore expect a satisfactory theory of crossed complexes over algebroids, say, where an algebroid is the name also used for an R-category where R is a commutative ring. But it is difficult to envisage a corresponding theory for Lie algebras, Jordan algebras, or for some other varieties of algebras. As an example of the difficulty, there is a standard functor assigning to an associative algebra a Lie algebra, with bracket $[x, y] = xy - yx$. It is not clear how this should be extended to the case of R-categories, but it seems likely that extra structure will be needed to do so. For example, Brown-Gilbert-Shrimpton, 1989, contains a suggestion for the extra structure required to define a commutator subgroupoid of a groupoid.

A further point is that the monoidal closed structure for crossed complexes over groupoids is deduced from the corresponding structure on ω-groupoids, as explained above. So the analogous theory for algebroids would need an equivalence between crossed complexes over algebroids and a form of ω-algebroids. The initial steps in such a theory have been taken in Mosa, 1987, but there are considerable technical difficulties in completing the task.

Crossed complexes are expected to be only the first step towards a truly non-Abelian homotopical and homological algebra. In particular, if the classifying space BC of a crossed complex C is simply connected, then it is up to homotopy just a product of Eilenberg-Mac Lane spaces. The theory was somewhat castigated in Grothendieck, 1983, for this limitation. Put in another viewpoint, crossed complexes (reduced and of free type) form the first level in the tower of homotopy theories of Baues, 1989a. Nonetheless, if one is going to attempt the whole theory, it is not unreasonable to start with a thorough study of the first case, as is also done by Baues. This case includes the theory of crossed modules, which completely describe homotopy 2-types. The algebraic and geometric implications of even this fact seem worthy of further study.

2 Problems on cat^n-groups

Recall from Loday, 1982, (but with the notation of Brown-Loday, 1987b) that a cat^1-group consists of a group G and two endomorphisms s, t of G such that $st = t, ts = s$, and $[Ker\ s, Ker\ t] = 1$. It is shown in loc. cit. that the category of cat^1-groups is equivalent to the category of crossed modules.

Recall further from Loday, 1982, that a cat^n-group G (there called an n-cat-group) consists of a group G , called the big group of G, together with $2n$ endomorphisms s_i, t_i of G, $i = 1, 2, \dots, n$, such that each $(G; s_i, t_i)$ is a cat^1-group, and the various such structures are compatible in the sense that if $i \neq j$ then

$$s_i s_j = s_j s_i, t_i t_j = t_j t_i, s_i t_j = t_j s_i.$$

It is shown in loc. cit. that the category of cat^2-groups is equivalent to a category of crossed squares. Such an object consists of a square of groups and morphisms

$$\begin{tikzcd} L \arrow[r, "\lambda"] \arrow[d, "\lambda'"'] & M \arrow[d, "\mu"] \\ N \arrow[r, "\nu"'] & P \end{tikzcd}$$

together with actions of P on the groups L, M, N, and a function $h : M \times N \to L$. The actions of P yield actions of M and N on M, N, L via P. The following axioms must hold.

2.1(i) The morphisms $\lambda, \lambda', \mu, \nu$ and $\kappa = \mu\lambda = \nu\lambda'$ are crossed modules and λ and λ' are P-equivariant.

2.1(ii) $h(mm', n) = h({}^m m', {}^m n)h(m, n)$, $h(m, nn') = h(m, n)h({}^n m, {}^n n')$;

2.1(iii) $\lambda h(m, n) = m\, {}^n m^{-1}$, $\lambda' h(m, n) = {}^m n n^{-1}$;

2.1(iv) $h(\lambda l, n) = l\, {}^n l^{-1}$, $h(m, \lambda' l) = {}^m l l^{-1}$;

2.1(v) $h({}^p m, {}^p n) = {}^p h(m, n)$;

for all $l \in L, m, m' \in M, n, n' \in N, p \in P$.

Morphisms of crossed squares are defined in the obvious way.

In the equivalence between crossed squares and cat^2-groups, the above crossed square corresponds to a cat^2-group with big group a repeated semidirect product $(L \rtimes M) \rtimes (N \rtimes P)$, which in fact is isomorphic to $(L \rtimes N) \rtimes (M \rtimes P)$. The h-function of the crossed square corresponds to the commutator in the big group.

This result has been generalised to an equivalence between cat^n-groups and a category of crossed n-cubes of groups in Ellis-Steiner, 1987.

Loday defines in loc. cit. (with different notation) functors

$$\Pi : (n\text{-cubes of spaces}) \to (\text{cat}^n\text{-groups}),$$

$$\mathsf{B} : (\text{cat}^n\text{-groups}) \to (n\text{-cubes of spaces}),$$

such that $\Pi\mathsf{B}$ is naturally equivalent to the identity. He also proves that a localisation of the category (cat^n-groups) is equivalent to the homotopy category of connected, pointed $(n+1)$-coconnected CW-complexes. In fact the composite of B with the obvious forgetful functor taking the final vertex of the cube yields the classifying space functor B. Then for a cat^n-group G the classifying space satisfies $\pi_i B\mathsf{G} = 0 for i > n+1$, and if X is any connected CW-complex such that $\pi_i X = 0$ for $i > n+1$, then there is a cat^n-group G such that $X \simeq B\mathsf{G}$. (One gap in the argument is filled by Steiner, 1986.)

The definition of the classifying space functor B is not hard. A cat^n-group can also be regarded as an n-fold groupoid in the category of groups. Taking the simplicial nerve in each of the n groupoid directions yields a simplicial group, whose classifying space is defined to be BG. The classifying space of a crossed module, i.e. the case $n = 1$ of this construction, is used in Loday, 1987.

The definition of the fundamental cat^n-group functor Π is quite subtle. A detailed account is given in Gilbert, 1988.

Brown-Loday, 1987a, prove a Generalised Van Kampen Theorem for the functor Π. This allows for some explicit computations. It would not be possible here to detail all the applications of the theorems and related algebra, since these amount to about 20 papers over the last three years in the areas of homotopy theory, algebraic K-theory, non-Abelian homology of groups, non-Abelian homology of Lie algebras, higher Hopf formulae, exact sequences in homology, and so on. But let us start with one result and problem.

Let $r : P \to R$ and $s : P \to S$ be morphisms of groups. Form the homotopy pushout diagram

$$\begin{array}{ccc} BP & \longrightarrow & BR \\ \downarrow & & \downarrow \\ BS & \longrightarrow & X \end{array} \qquad (7)$$

induced by r and s, where B is the classifying space functor on groups. If r and s are injective, then X is a $K(\pi, 1)$, as was proved by Whitehead, 1939. If r and s are surjective with kernels M and N respectively, then the 3-type of X is given by the universal crossed square

$$\begin{array}{ccc} M \otimes N & \longrightarrow & M \\ \downarrow & & \downarrow \\ N & \longrightarrow & P \end{array} \qquad (8)$$

where $M \otimes N$ is a non-Abelian tensor product (Brown-Loday, 1987a), and the h-map $M \times N \to M \otimes N$ is the universal biderivation (cf property 2.1(ii) of a crossed square). If P is finite, then so also is $M \otimes N$, as was proved in Ellis, 1987b.

This description of the 3-type of the homotopy pushout X is a consequence, with some extra work, of results of Loday, 1982, and the GVKT for the fundamental crossed square of a square of spaces (Brown-Loday, 1987a). It is quite satisfactory as far as it goes. It implies of course complete descriptions of $\pi_2 X$ and $\pi_3 X$, the first as $(M \cap N)/[M, N]$, and the second as the kernel of the commutator morphism $M \otimes N \to P$, $m \otimes n \mapsto mnmn$. A number of calculations of $M \otimes M$ have been done, so there is a lot of information on this case (Brown-Johnson-Robertson, 1987, Aboughazi, 1987, Johnson, 1987a,b, Gilbert, 1987, Gilbert-Higgins, 1989).

By contrast, the Postnikov description of the 3-type of X would require an element $k^3 \in H^3(P/MN, (M \cap N)/[M, N])$ which would then allow one to construct a space $X^{(2)}$; one then needs an element $k^4 \in H^4(X^{(2)}, Ker(M \otimes N \to P))$. It is not clear that such a description is practicable or would be helpful in this case, particularly as one seems to need to describe the above cohomology groups, and so in effect also a load of other spaces, before one can describe this particular space.

However there are still problems with the next stage.

Problem 2.2. *Find an algebraic description of $\pi_4 X$ in this case.*

The above formula for $\pi_2 X$ was first proved in Brown, 1984a, as a consequence of the Van Kampen Theorem for maps. An alternative proof has been given by R.Fenn in 1985 (unpublished) using methods of transversality analogous to those used by Whitehead, 1941, 1946, 1949, in proving his description of $\pi_2(Y \cup \{e^2_\lambda\}, Y)$ as a free crossed $\pi_1 Y$-module. However the above description of $\pi_3 X$ has not been recovered by transversality methods. Can one move in the opposite direction and apply these algebraic methods to transversality questions?

If $r : P \to R$ is surjective with kernel M, then it is possible to describe the 2-type of the homotopy pushout X as given by the crossed module $s_* M \to S$ *induced* from the inclusion crossed module $M \to P$ by the morphism $s : P \to S$. This follows from results of Loday, 1982, and the GVKT for maps proved in Brown-Higgins, 1978, which also contains alternative presentations of induced crossed modules.

Problem 2.3. *Find an algebraic description of the 3-type of X in this case.*

It may be shown (Brown, 1989b) that any simply-connected 3-type (with finitely generated π_2 and π_3) may be described by a crossed square as above in which $M = N = P$ and the groups M and L are abelian. This follows from the known description of such 3-types as determined by a quadratic map $\pi_2 \to \pi_3$.

Problem 2.4. *Is any simply connected n-type describable by a crossed n-cube of Abelian groups?*

If this were so, the algebraic calculations with such crossed n-cubes could prove much easier, and would lend themselves better to such constructions as, for example, localisation.

It seems remarkable to the writer that a description of n-types can be given in so few lines: essentially, it says that n-types are described by n-fold groupoids. What is not so clear is how useful this description could be for say the n-types of spheres.

Problem 2.5. *Try and find a description of the 6-type of S^3 in terms of a crossed 5-cube of finitely generated abelian groups.*

I have no idea if this is possible. All I am saying is that the ideal scenario for this topic would be the existence and utility of such descriptions.

It would also be interesting to use crossed squares and the GVKT to find results on the 3-type of the complements of knotted 2-spheres in 4-spheres, as has been done by different methods for 2-types in Plotnick-Suciu, 1985.

Loday, 1982, derives from a cat^n-group G a non-Abelian chain complex whose homology is the homotopy of $B\mathsf{G}$.

Problem 2.6. *Give descriptions of Whitehead products and other homotopy constructions, such as Hopf invariants, and Toda brackets, in terms of this chain complex.*

This has been solved completely for $n = 2$, but for no other case. However the case $n = 2$ does lead to a natty description of the Whitehead products $\pi_2 \times \pi_2 \to \pi_3$ in the space SBP for a group P. In terms of the results that $\pi_2 = P^{ab}$, and π_3 is given by the kernel of $P \otimes P \to P$, the Whitehead product is induced by the map $P \times P \to P \otimes P, (m, n) \mapsto (n \otimes m)(m \otimes n)$, and so is computable in many specific cases.

Conduché, 1984, has given models of 3-types in terms of crossed modules of length 2, which have been used in other situations (Grandjean-Vale, 1986, 1988). Baues, 1989b, has given models of 3-types and 4-types in terms of what he calls quadratic chain complexes. His models are more directly related to homological invariants and do enable some striking calculations.

Problem 2.7. *Explore the relations between quadratic chain complexes, crossed modules of length 2, and crossed squares.*

[Work on this has been done by Ellis [21].]

The general relations between crossed n-cubes of groups and simplicial groups is currently under study by Porter [22]. Also Carrasco-Cegarra, 1989, gives a generalisation of the Dold-Kan theorem to an equivalence between simplicial groups and a non-Abelian chain complex with a lot of extra structure, generalising Conduché's crossed modules of length 2. However, at present the only categories which have been shown to be the image category for a Generalised Van Kampen Theorem are: the category of cat^n-groups; its equivalent category of crossed n-cubes of groups; and what are near to specialisations of these, such as the category of crossed complexes, and categories equivalent to that one.

The above description of the third homotopy group $\pi_3 SBP$ as $Ker(P \otimes P \to P)$ should in principle be obtainable from the spectral sequence of a bisimplicial space. In practice, as was observed by Zisman (private communication), such an approach yields a chain complex of groups involving free products of P with itself, and it is not clear how to deduce the explicit formula for $\pi_3 SBP$ which pops out of the GVKT for squares of spaces. This suggests the algebraic value of cat^n-groups and crossed n-cubes of groups.

Problem 2.8. *Pursue this apparent advantage into other applications which have previously used simplicial groups.*

One of the irritations of the theory of cat^n-groups and the fundamental cat^n-group is that it is still a theory for pointed spaces. However, crossed complexes allow an arbitrary set of base points, and this is crucial in the algebra and applications. For example, the internal hom $CRS(B, C)$ for crossed complexes B, C has vertex set the morphisms $B \to C$ of crossed complexes. This echoes one of the advantages of groupoids over groups, namely that groupoids form a cartesian closed category. Another advantage of groupoids is that there is a useful notion of covering groupoid, modelling the notion of covering space (see Brown, 1988b). This has a generalisation to a convenient notion of fibration, and also covering morphism, of crossed complexes (Howie, 1979) which is applied in Brown-Golasinski, 1989, to the homotopy theory of crossed complexes.

Problem 2.9. *Give a version of cat^n-groups and the fundamental cat^n-group for a set of base points, and investigate the applications to covering spaces.*

The existence of the monoidal closed structure on the category of crossed complexes is useful in the applications, as mentioned in section 2. No such structure is available at present for cat^n-groups, or for cat^n-groupoids, whatever those might be. Indeed, it is difficult to formulate a rule analogous to

$$[\pi X_*, C] \cong [X, BC]$$

since we do not have a notion of the structure X_* on X which would be required to define an appropriate πX_*, nor do we have an appropriate notion of homotopy for cat^n-groups, except in the sense of a localisation with respect to weak equivalence. By contrast, the category of crossed complexes has a Quillen model structure (Brown-Golasinski, 1989, Baues, 1989b) in which the class of cofibrant crossed complexes includes the crossed complexes which are of free type, for example the fundamental crossed complexes of CW-complexes with the skeletal filtration.

Problem 2.10. *Investigate Quillen model structures on the category of cat^n-groups, and give an explicit description of a useful notion of cofibrant cat^n-group.*

On the other hand, Baues, 1989b, does give conditions for a homotopy classification of the form $[X, Y] \cong [\pi X, \pi Y]$ where πX is a quadratic chain complex assigned to the CW-complex X.

The methods of crossed complexes, ω-groupoids and cat^n-groups have not at present been much related to some other general trends in homotopy theory, for example stable homotopy theory, principally because of the work that has been required to develop the former methods, in the first instance on their own terms.

Problem 2.11. *Relate the algebraic methods of ω-groupoids and cat^n-groups to methods in stable homotopy theory, such as the little cubes operads.*

Another standard method in homotopy theory is that of localisation.

Problem 2.12. *Develop a theory of localisation of cat^n-groups, and in particular prove a localised version of the Generalised Van Kampen Theorem.*

The last proposal looks hard or even unlikely. However a start on the localisation of crossed modules has been made in Korkes, 1987, and a series of preprints by Korkes-Porter.

As explained before Problem 1.11, the work of Whitehead, 1948, Lue, 1979, Norrie, 1986, 1987a,b and Brown-Gilbert, 1989, on the automorphism structures for crossed modules shows that the automorphisms of a crossed module form part of a structure which is an algebraic model of homotopy 3-types, for example a crossed square, a braided crossed module, or a simplicial group whose Moore complex is of length 2. One interest of this is to give a way in which such structures arise from purely algebraic considerations.

Problem 2.13. *Show that the automorphism group of a cat^n-group forms a part of the structure of a cat^{n+1}-group.*

A solution to this would show a correspondence between homotopy theory and notions of higher order symmetry.

Crossed modules are intimately involved in notions of non-Abelian homological algebra (see the survey article Brown, 1984a). In particular, free crossed modules over a free group give an expression for identities among relations (see the survey article Brown-Huebschmann, 1982). Suppose now that $\mathcal{P} = (X; R)$ and $\mathcal{Q} = (X; S)$ are two presentations of the same group G with the same set X of generators. Then we can form two 2-complexes $K(\mathcal{P})$ and $K(\mathcal{Q})$ corresponding to these presentations. Let K^1 be the common 1-skeleton of these. A consequence of the GVKT for squares of spaces is that the triad homotopy group

$$\pi_3(K(\mathcal{P}) \cup K(\mathcal{Q}); K(\mathcal{P}), K(\mathcal{Q}))$$

is isomorphic to the non-Abelian tensor product

$$\pi_2(K(\mathcal{P}), K) \otimes \pi_2(K(\mathcal{Q}), K).$$

Problem 2.14. *Find an algebraic explanation of this involvement of the non-Abelian tensor product, in terms of an algebraic analysis of identities among sets of relations, analogous to that which shows the applicability of free crossed modules over free groups to the description of identities among relations.*

One would also hope that the non-Abelian tensor product would have applications to some of the classical problems in combinatorial group theory (the Whitehead conjecture, which is discussed from a more general viewpoint in Gilbert-Higgins, 1989; the Andrews-Curtiss conjecture; and the Kervaire-Laudenbach conjecture).

A useful technique for dealing with presentations is the Fox free differential calculus. The essence of this was described in Whitehead, 1949, in terms of the relations between the fundamental crossed complex of a CW-complex X and the cellular chain complex of the universal cover of X. This has been generalised in Brown-Higgins, 1989a, to a functor

$$\Delta : (\text{crossed complexes}) \to (\text{chain complexes with a groupoid as operators}) .$$

Problem 2.15. *Is there a version of the latter category and of this functor but with domain category that of crossed squares or crossed n-cubes of groups?*

If M and N are normal subgroups of a group P, then one may define a non-Abelian exterior product $M \wedge N$ as the quotient of $M \otimes N$ by the relations $m \otimes m = 1$ for $m \in M \cap N$. This exterior product occurs in eight-term exact sequences in the integral homology of groups (Brown-Loday, 1987a, Ellis, 1987a). A version of this sequence but for mod q homology has been given by Ellis-Rodriguez Fernandez, 1989.

However there are in the literature ten-term exact sequences, with terms not so explicitly described, but which are valid for homology with coefficients in a module.

Problem 2.16. *Give formulations of the eight-term exact sequences involving the exterior product, but also involving coefficients in a non-trivial module.*

Here is a basic homotopical question. Cohomology operations may be described as maps $K(A, m) \to K(B, m + r)$ or as cohomology classes.

Problem 2.17. *Find a representation of the $(m + r)$-type of a $K(A, m)$ as a crossed $(m+r)$-cube so as to obtain explicit representations of the elements of $H^{m+r}(K(A, m), B)$.*

This may not be possible since the homotopy category of cat^n-groups is obtained by a localisation process. The theory may have its main use only in those cases where the geometry happens to give rise to explicit representations of the spaces and maps concerned in the situation.

One motivation for the manuscript Grothendieck, 1983, was the search for an appropriate non-Abelian cohomology whose coefficients would be a homotopy type. The particular models investigated there are kinds of coherent ∞-categories, which are intended to give higher dimensional versions of Picard groupoids. Omitted from the investigation is the possible implications of the fact that n-types may be modelled by n-fold groupoids. The aim of investigating basic structures in homotopy theory is not only to answer some homotopical questions, but also to produce methods which have some prospect of wider applicability.

I should add that this paper represents a personal view, with a number of avenues and references unexplored. I hope that time will allow judgement on the relative values of these problems, and so give pointers to further work.

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3 Appendix: Later results

3.1 Problem 1.1

This has now been solved in the thesis of Andy Tonks [27]. He uses Artin-Mazur diagonals to prove:

Proposition 3.1 *Given simplicial sets K, L, the crossed complex homomorphism*

$$\pi(K \times L) \xrightarrow{a_{K,L}} \pi K \otimes \pi L$$

is given by the homomorphism which acts on the generators of $\pi(K \times L)$ by

$$\begin{aligned}
(x_0, y_0) &\mapsto x_0 \otimes y_0 \\
(x_1, y_1) &\mapsto d_1 x_1 \otimes y_1 \;\cdot\; x_1 \otimes d_0 y_1 \\
(x_2, y_2) &\mapsto (d_2 x_2 \otimes d_0 y_2)^{d_0 x_2 \otimes d_0^2 y_2} \;\cdot\; x_2 \otimes d_0^2 y_2 \;\cdot\; (d_1 d_2 x_2 \otimes y_2)^{d_1 x_2 \otimes d_0^2 y_2} \\
(x_n, y_n) &\mapsto \prod_{i=0}^{n} (d_{i+1}^{n-i} x_n \otimes d_0^i y_n)^{c_i(x_n) \otimes d_0^n y_n}
\end{aligned}$$

where $c_i(x)$ is given by the one cell $d_0^i d_{i+1}^{n-i-1} x$ or by the identity at $d_0^n x$ if $i = n$.

The associativity properties of this morphism and its homotopy inverse are crucial for the homotopy coherence properties used in the constructions in [9].

However the general problem of a Homological Perturbation Theory for crossed complexes is still unsolved.

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