

Topological groupoids: I. universal constructions

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1. Introduction

The notions of topological and of Lie groupoid were introduced by EMMERMANN in a series of papers for applications to differential topology and geometry. More recent interest in topological groupoids has come from the construction for a topological groupoid T of a classifying space B_T [A 17, A 52, A 53] and the use of B_T for classifying foliations and constructing characteristic classes of foliations while further interest in Lie groupoids comes from applications to Lie equations [A 29]. However it is noticeable that many of the basic constructions for topological groups or abstract groupoids have not been developed for topological groupoids.

One example of this is quotienting. For topological groups this is a triviality — if N is a normal subgroup of the topological group G , the identification mapping $G \rightarrow G/N$ is an open mapping, and this makes it easy to prove continuity of the composition in G/N . For a topological groupoid G and normal subgroupoid N , the abstract quotient G/N defined by HIGGINS [7] can be given the identification topology, but the proof of continuity used in the group case breaks down, so we present a different construction of a topology in G/N making it a topological groupoid with the appropriate universal property; we are able to give some sufficient conditions for this topology to agree with the identification topology.

This apparent disadvantage of topological groupoids has compensations. We generalise the notion of quotient to that of forming the topological groupoid G/R where R is a set of relations on G (i.e. R is a subset of $G \times G$). This construction then generalises not only the quotient, but gives a topological version of HIGGINS' universal groupoids $U_c(G)$ [7]. A special case of this construction is that of the universal topological group of a topological groupoid, which itself specialises to free topological groups [5, 17, 15, 16] and free products of topological groups [6, 11, 12, 13], thus solving a well-known problem [6] of giving one construction containing the last two.

An alternative construction to those given here could be obtained by working in a convenient category of topological spaces for which the pull-back of identification maps is an identification map. However the methods given here apply to the ordinary topological category as well as to convenient ones.

In order to give an idea of the scope of applications of topological and Lie groupoids, we have appended a Bibliography on this topic—we are grateful to J. VIKSNIK for help in preparing this.

A sequel to this paper will deal with G -spaces and topological covering morphisms.

2. Main results

Our terminology for categories follows that of [8], except that

- (i) The large category of all small categories is written \mathcal{C} and the large category of all small groupoids is written \mathcal{S} .
- (ii) A category whose only arrows are identities is called *pointlike*—the term discrete used in [8] and elsewhere would cause confusion for topological categories.
- (iii) If C is a category, we write $\text{Ob}(C)$ for its class of objects and $a \in C$ means a is an arrow of C .
- (iv) The domain and codomain maps are written δ' and δ respectively.

Our main concern is with the category \mathcal{TS} of topological groupoids. However the methods give easily results on topological categories, and so we consider also the category \mathcal{TC} of topological categories. The definitions are as follows.

A *topological category* C over X is a small category with topologies on its object set $X = \text{Ob}(C)$ and its set of arrows such that the structure functions δ' , δ , id and \circ ([8] p. 10) are continuous on their domains. A *topological groupoid* G is a topological category whose underlying category is a groupoid, and such that the inverse function $a \mapsto a^{-1}$ is continuous. A *morphism* of topological categories (groupoids) is a morphism of categories, i.e. a functor, which is continuous on both objects and arrows.

The following examples of topological categories and groupoids are well known, but are given here as they are essential later.

Example 1. A topological group can be regarded as a topological groupoid with only one object. The disjoint union of topological groups is a topological groupoid which is not a topological group.

Example 2. A topological groupoid G is *pointlike* if the only elements of G are identities. In this case we identify G with $\text{Ob}(G)$. If X is a topological space, we write X also for the essentially unique, pointlike topological groupoid with object space X . Notice that X is an initial object in the category \mathcal{TS}/X whose objects are topological groupoids over X and whose morphisms are morphisms of \mathcal{TS} which are the identity on objects.

Example 3. Let X be a topological space. Then X determines a topological groupoid over X whose set of arrows is $X \times X$ and with composition given by $(y, z) \cdot (x, y) = (x, z)$. This groupoid is written $X \times X$, and any object of \mathcal{TS}/X isomorphic in \mathcal{TS}/X to $X \times X$ is called a *tree topological groupoid* over X . Clearly $X \times X$ is a terminal object in the category \mathcal{TS}/X of the previous example.

Example 4. Let $\{C_i\}_{i \in J}$ be a family of topological categories. Then their *product* topological category $C = \prod_{i \in J} C_i$ is easily defined using the product topology on both objects and arrows. If each C_i is a topological groupoid, so also is C . Similarly the *coproduct* (or *disjoint union*) $\bigsqcup_{i \in J} C_i$ is well-defined, and is a topological groupoid if and only if each C_i is a topological groupoid.

Example 5. (EKRESMANN) A topological groupoid G over X is *globally trivial* if there is an x_0 in X and a continuous function $\lambda: X \rightarrow G$ such that $\lambda(x) \in G(x_0, x)$ for all x in X . It is easy to see that G is globally trivial if and only if G is isomorphic in \mathcal{TS}/X to $G\{x_0\} \times (X \times X)$, where $G\{x_0\}$ is the group $G(x, x)$. (Locally trivial groupoids will be discussed in a sequel.)

Example 6. Let C be a topological category. Then a subcategory D of C with its subspace topology is a topological category. If C is a topological groupoid and D is a subgroupoid of C then D with its subspace topology is a topological groupoid. It follows from this that equalisers exist in both \mathcal{TE} and \mathcal{TS} , and from this and Example 4 that these categories are complete. One of the aims of this paper is to prove these categories cocomplete.

Example 7. If C is an abstract category, then we can form two topological categories $R_{\mathcal{E}}(C), L_{\mathcal{E}}(C)$ by using respectively the indiscrete and discrete topologies on both objects and arrows. In fact $L_{\mathcal{E}}, R_{\mathcal{E}}$ are functors which are respectively left and right adjoint to the forgetful functor $F_{\mathcal{E}}: \mathcal{TE} \rightarrow \mathcal{E}$, and a consequence is that $F_{\mathcal{E}}$ preserves both limits and colimits. Similarly we have functors $L_{\mathcal{S}}, R_{\mathcal{S}}; \mathcal{S} \rightarrow \mathcal{TS}$ which are left and right adjoint to the forgetful functor $F_{\mathcal{S}}: \mathcal{TS} \rightarrow \mathcal{S}$.

We now come to our main definitions. Let C be a topological category, and R any subset of $C \times C$ — we wish to define and then construct, a topological category C/R in which the relations $a \sim b$ for (a, b) in R shall hold. This is easily defined by a universal property, but for the applications we want a more general situation in which we insist that C/R shall be an object of some subcategory \mathcal{F} of \mathcal{TE} . Two useful cases are (i) $\mathcal{F} = \mathcal{TE}$; here C/R is simply the topological category C with relations R , and (ii) $\mathcal{F} \neq \mathcal{TE}$ but R is empty; then the \mathcal{F} -category C with relations R is an object $C_{\mathcal{F}}$ of \mathcal{F} , and $C \rightarrow C_{\mathcal{F}}$ is a reflection from \mathcal{TE} to \mathcal{F} . Typical applications of the second case are when $\mathcal{F} = \mathcal{TS}$, and when \mathcal{F} is the category of HAUSDORFF topological categories (or groupoids).

Definitions. Let \mathcal{F} be a subcategory of \mathcal{TE} , let C be any topological category and R any subset of $C \times C$. The \mathcal{F} -category C with relations R consists of an object $C|_{\mathcal{F}}R$ of \mathcal{F} , and a morphism $p: C \rightarrow C|_{\mathcal{F}}R$ of topological categories such that

1. If $(a, b) \in R$, then $p(a) = p(b)$.
2. If K is an object of \mathcal{F} and $f: C \rightarrow K$ is a morphism of topological categories such that $f(a) = f(b)$ for all (a, b) in R , then there is a unique morphism $f^*: C|_{\mathcal{F}}R \rightarrow K$ in \mathcal{F} such that $f^*p = f$.

If such a \mathcal{F} -category C with relations R exists for all objects C of $\mathcal{F}\mathcal{E}$ and subsets R of $C \times C$, then we say $\mathcal{F}\mathcal{E}$ admits arbitrary relators with values in \mathcal{F} . In the case $\mathcal{F} = \mathcal{F}\mathcal{E}$, this is abbreviated to $\mathcal{F}\mathcal{E}$ admits arbitrary relators, and $C|_{\mathcal{F}}R$ is abbreviated to $C|R$.

Similar definitions apply to the category $\mathcal{F}\mathcal{S}$.

Some conditions on \mathcal{F} are clearly going to be needed for arbitrary relators with values in \mathcal{F} to exist.

Let \mathcal{F} be a full subcategory of $\mathcal{F}\mathcal{E}$. We say \mathcal{F} admits arbitrary subobjects if for any object C of \mathcal{F} any subcategory C' of C with its subspace topology is also an object of \mathcal{F} . If \mathcal{F} is a full subcategory of $\mathcal{F}\mathcal{S}$, then this condition is required to hold only for subgroupoids of objects of \mathcal{F} .

Our main theorem is:

Theorem 1. *Let \mathcal{F} be a full, non-empty subcategory of $\mathcal{F}\mathcal{E}$ such that \mathcal{F} admits arbitrary subobjects and is closed under the formation of arbitrary products. Then $\mathcal{F}\mathcal{E}$ admits arbitrary relators with values in \mathcal{F} .*

The proof will be given in § 4.

Corollary 2. *The category $\mathcal{F}\mathcal{E}$ admits arbitrary relators. This is the case $\mathcal{F} = \mathcal{F}\mathcal{E}$ of Theorem 1.*

We shall need to know that if C is a topological groupoid, then so also is the topological category $C|R$.

Let C, R, \mathcal{F} satisfy the conditions of Theorem 1.

Proposition 3. *If \mathcal{F} includes D^{op} for every object D of \mathcal{F} , and C is a topological groupoid, then so also is $C|_{\mathcal{F}}R$. Consequently the category $\mathcal{F}\mathcal{E}$ admits arbitrary relators. The proof is given in § 4. The last part of the Proposition also follows from Theorem 1 with $\mathcal{F} = \mathcal{F}\mathcal{S}$.*

We now show that the above results imply the well known result that abstract categories admit arbitrary relators. More generally, let \mathcal{F} be a full subcategory of \mathcal{E} ; then we can define in an obvious way what is meant by \mathcal{F} being closed under arbitrary subobjects and arbitrary products (as usual, when the objects of \mathcal{F} are groupoids, then by a subobject is meant a subgroupoid).

Corollary 4. *If \mathcal{F} admits arbitrary subobjects and products, then \mathcal{E} admits arbitrary relators with values in \mathcal{F} .*

The proof is given in § 4.

We would also like to determine in the topological case the underlying category of the \mathcal{F} -category C with relations R .

Let C be a topological category and R a subset of $C \times C$. Let \mathcal{F} satisfy the conditions of Theorem 1, and let \mathcal{F} be the full subcategory of \mathcal{E} on the underlying categories of the objects of \mathcal{F} .

Proposition 5. *If $R_{\mathcal{F}}(\mathcal{F}) \subseteq \mathcal{F}$, then there is an isomorphism of \mathcal{F} -categories*

$$\mathbf{F}_{\mathcal{F}}(C|_{\mathcal{F}}R) = \mathbf{F}_{\mathcal{F}}(C)|_{\mathcal{F}}R.$$

Proof. It may be checked that the two morphisms

$$R_{\mathcal{C}} F_{\mathcal{C}}(C) \rightarrow R_{\mathcal{C}} F_{\mathcal{C}}(C/\mathcal{I}R)$$

$$R_{\mathcal{C}} F_{\mathcal{C}}(C) \rightarrow R_{\mathcal{C}}(F_{\mathcal{C}}(C)/\mathcal{I}R)$$

both define the \mathcal{I} -category $R_{\mathcal{C}} F_{\mathcal{C}}(C)$ with relations R . The result follows easily.

Propositions 5 may not be true without some condition on \mathcal{I} such as that given. For example if \mathcal{I} consists of all HAUSDORFF topological categories, then $C/\mathcal{I}R$ will be HAUSDORFF; so that if C is a non-HAUSDORFF, pointlike category, and R is empty, then the projection $C \rightarrow C/\mathcal{I}R$ will not be a bijection of sets.

Corollary 6. *Let G be a topological category whose underlying category is a groupoid. Then G admits a coarser topology making it a topological groupoid G' and such that the morphism $G \rightarrow G'$ is universal for maps of G to topological groupoids.*

In fact G' can be taken to be the $\mathcal{I}\mathcal{S}$ -category G with empty relations—Proposition 5 shows that G and G' have the same underlying groupoid.

3. Applications

3.1. Quotient groupoids. Let N be a wide normal subgroupoid of a topological groupoid G (by which is meant that for any $x, y \in \text{Ob}(N)$ and $a \in G(x, y)$, $aN\{x\} = N\{y\}a$). We say arrows a, b of G bridge the same components of N if there are arrows τ, τ' of N such that $b = \tau'ax$. Let us define an equivalence relation R on G by $(a, b) \in R$ if and only if $a = b$ or a, b bridge the same components of N . Then the topological groupoid G with relations R is well-defined; it is called the *quotient topological groupoid* and is written G/N .

It is clear that $p: G \rightarrow G/N$ satisfies the universal property: if $f: G \rightarrow H$ is any morphism of topological groupoids such that $f(N)$ is pointlike, then there is a unique morphism $f^*: G/N \rightarrow H$ such that $f^*p = f$. From Proposition 5 (with $\mathcal{I} = \mathcal{I}\mathcal{S}$, $\mathcal{I} = \mathcal{S}$) it follows that the underlying groupoid of G/N is the quotient [7] of the underlying groupoid of G .

In the case G is a topological group, the identification topology on G/N makes G/N a topological group, and so the above topology and the identification topology coincide. For the groupoid case we have only a partial result.

Proposition 7. *Let G be a locally compact, HAUSDORFF topological groupoid and N a compact normal subgroupoid. Then the quotient morphism $p: G \rightarrow G/N$ is an identification map.*

The proof is given in § 4.

Remarks. 1. We have no example for which $p: G \rightarrow G/N$ can be shown not to be an identification map.

2. However $\text{Ob}(p): \text{Ob}(G) \rightarrow \text{Ob}(G/N)$ is an identification map. For suppose $f: \text{Ob}(G/N) \rightarrow Y$ is such that $f \circ \text{Ob}(p)$ is continuous. Define $g: G \rightarrow Y \times Y$ by

$q \rightarrow (f_1 p_1^{-1} \circ g_1, f_2 p_2^{-1} \circ g_2)$. Then q is a morphism of topological groupoids which annihilates \mathcal{K} . Hence $q = p_2^{-1} p_1$ and so $f = (f_2 \circ p_2^{-1} p_1)$ is continuous.

3.2. Coequalizers and colimits. Let \mathcal{F} be any full subcategory of the entire of all topological categories such that \mathcal{F} admits arbitrary subobjects and products.

An easy consequence of Theorem 1 is that \mathcal{F} admits arbitrary coequalizers. These can, as is well known, be defined by relations $(g, h), (a, b)$ for a and b in B , where $\{g, h\} \in \mathcal{F}$ is the family of morphisms to B .

If \mathcal{F} also admits arbitrary coproducts, then the existence of coequalizers in \mathcal{F} implies that \mathcal{F} is cocomplete. Similar remarks apply to the case of products so we have:

Corollary 8. The categories $\mathcal{F} \mathcal{C}$ and $\mathcal{F} \mathcal{S}$ are cocomplete.

A particular case of Corollary 8 is that arbitrary coproducts exist in $\mathcal{F} \mathcal{S}$ and $\mathcal{F} \mathcal{C}$.

The inclusion $\mathcal{I}: \mathcal{F} \mathcal{S} \rightarrow \mathcal{F} \mathcal{C}$ preserves coproducts (Proposition 3) and hence preserves colimits. The inclusion $\mathcal{J}: \mathcal{F} \mathcal{C} \rightarrow \mathcal{F} \mathcal{S}$ preserves colimits of topological groupoids is also true in $\mathcal{F} \mathcal{C}$.

3.3. Categories of fractions.

Then we can "make the elements of a topological category (groupoid) invertible" by forming a category of fractions. Let \mathcal{C} be a topological category (groupoid) and let \mathcal{S} be a subcategory of \mathcal{C} such that \mathcal{S} is a topological groupoid and \mathcal{S} is dense in \mathcal{C} . Then we can form a quotient category $\mathcal{C}[\mathcal{S}^{-1}]$.

...
... since
... f, k and l
... coequalizer
... implies
... topological groupoids

...
... exist in both $\mathcal{F} \mathcal{C}$

... and coequalizers (by Prop. ...)
... a commutative square of ...
... in $\mathcal{F} \mathcal{S} \mathcal{K}$ and only if \mathcal{K} is a product.

... be any subset of a topological category \mathcal{C} .
... of \mathcal{S} invertible" as follows. Let $\mathcal{C}[\mathcal{S}^{-1}]$ be the ...
... with two objects $0, 1$, exactly one non-identity ...
... discrete topology on objects and arrows. Then we ...
... $\mathcal{F} \mathcal{C}$



i is the obvious inclusion morphism and j on the surround determined by ...
... i to u . Clearly if $u \in \mathcal{S}$, then $pi(u)$ has an inverse in $\mathcal{C}[\mathcal{S}^{-1}]$. It is easily checked ...
... that $p: \mathcal{C} \rightarrow \mathcal{C}[\mathcal{S}^{-1}]$ is universal for morphisms $f: \mathcal{C} \rightarrow \mathcal{D}$ such that $f(u)$ is invertible ...
... for all $u \in \mathcal{S}$. By Proposition 5, the underlying category of $\mathcal{C}[\mathcal{S}^{-1}]$ is the category of ...
... fractions constructed in [4].

A special case of this construction is when $\mathcal{S} = \mathcal{C}$; then $\mathcal{C}[\mathcal{S}^{-1}]$ is a topological ...
... category which is also a groupoid, but it is not clear that $\mathcal{C}[\mathcal{S}^{-1}]$ is a topological ...
... groupoid. However by Corollary 6 we can give $\mathcal{C}[\mathcal{S}^{-1}]$ a weaker topology for which ...
... it becomes a topological groupoid which we write $\mathcal{G}(\mathcal{C})$. Clearly the morphism ...
... $\mathcal{C} \rightarrow \mathcal{G}(\mathcal{C})$ is universal for morphisms of \mathcal{C} into topological groupoids.

Further \mathcal{G} defines a functor $\mathcal{F} \mathcal{C} \rightarrow \mathcal{F} \mathcal{S}$ which is left adjoint to the inclusion ...
... $\mathcal{I}: \mathcal{F} \mathcal{S} \rightarrow \mathcal{F} \mathcal{C}$. A corollary is that \mathcal{G} preserves colimits and \mathcal{I} preserves limits.

3.4. Universal categories and groupoids. Let \mathcal{C} be a topological category, ...
... and let $\sigma: \mathcal{O}(\mathcal{C}) \rightarrow \mathcal{X}$ be a continuous function. As stated in Example 2, we



...
...

identify $\text{Ob}(C)$ with the pointlike groupoid on $\text{Ob}(C)$, so that we have an inclusion morphism $i: \text{Ob}(C) \rightarrow C$. By 3.2, we can form the pushout

$$\begin{array}{ccc} \text{Ob}(C) & \xrightarrow{\sigma} & X \\ \downarrow i & & \downarrow i^* \\ C & \xrightarrow{\sigma^*} & U_\sigma(C) \end{array}$$

Proposition 9. $\text{Ob}(i^*)$ is a homeomorphism.

The proof will be given in § 4.

By virtue of Proposition 9, we may without loss assume that $U_\sigma(C)$ is a topological category over X . We refer to $\sigma^*: C \rightarrow U_\sigma(C)$ as the *universal morphism* determined by σ . Note also that by 3.2, if C is a topological groupoid, then so also is $U_\sigma(C)$.

Corollary 4 shows that we have given another proof of the existence of universal categories and groupoids in the abstract case [7]. Proposition 5 shows that the underlying category of $U_\sigma(C)$ is exactly that given for the abstract case. This implies that σ^* is injective on the non-identity arrows of C .

3.5. The universal topological group of a topological groupoid. Let G be a topological groupoid, and let $\sigma: \text{Ob}(G) \rightarrow P$ be the unique function to a singleton P . Then $U_\sigma(G)$ is a topological group called the *universal topological group* of G ; we write UG for $U_\sigma(G)$. Clearly $G \rightarrow UG$ is universal for morphisms of G into topological groups.

3.6. Applications to topological groups. We indicate briefly how the above constructions apply to give free topological groups and free products of topological groups.

Let X be a topological space, also regarded as a pointlike topological groupoid. Let $FM(X)$ be the universal group of the groupoid $X \times \mathbf{2}'$, and let $f: X \rightarrow FM(X)$ be the continuous mapping $x \rightarrow \sigma^*(x, 0)$. Then f is universal for continuous mappings of X into topological groups, (as is verified exactly as in the abstract case [3, 7]) and $FM(X)$ is the free MARKOV topological group on X [9].

The GRAEV free topological group $FG(X)$ [5] is defined for a space X with base point e , and is obtained from $FM(X)$ by adding the relation $f(e) = 1$. The composite $X \xrightarrow{f} FM(X) \rightarrow FG(X)$ is then universal for continuous mappings into topological groups which map e to the identity. The following gives an alternative description of $FG(X)$.

Proposition 9. If X is a topological space with base point, then $U(X \times X)$ is isomorphic as topological group to $FG(X)$.

A proof is given in § 4.

Finally, the free product of topological groups is simply the universal group of their disjoint union.

The generalizations of these constructions to free topological groups and free products of topological groups will be dealt with elsewhere, when it will be used to prove that open subgroups of free topological groups on \aleph_1 Hausdorff spaces are free topological groups [A 2].

Note also that similar methods apply to construct free topological and free products of topological manifolds.

4. Proofs

For the proof of Theorem 1 we need some preliminary \mathcal{F} - \mathcal{B} -morphisms (\mathcal{F} - \mathcal{B} -morphisms) $f: C \rightarrow D$ is an proper subcategory (subcategory) of \mathcal{D} iff clearly if $f: C \rightarrow D$ is any \mathcal{F} - \mathcal{B} -morphism subcategory (subcategory) \mathcal{D} of \mathcal{D} such generates \mathcal{D} .

The cardinality of a set X is $|X|$.

4.1. Lemma. Let $f: C \rightarrow D$.

Then $|D| \leq \max\{|C|, \aleph_1\}$.

The proof is similar to

4.2. Lemma. Let \mathcal{F}

collection of topological

has cardinality κ

The proof

4.3. Pr

category

by a

any results.
to generate \mathcal{D} if there exists
any $f: C \rightarrow D$.
 \mathcal{F} -morphisms) then there exists a
it the restriction of f mapping $C \rightarrow D$
in \mathcal{D} .
a \mathcal{F} - \mathcal{B} -morphism (\mathcal{F} - \mathcal{B} -morphism) generating \mathcal{D} .

of Lemma 1 in [1] and is omitted.

any subcategory of \mathcal{F} - \mathcal{B} - \mathcal{C} . Let A be a graph. Then the
category (groupoid) \mathcal{C} in \mathcal{F} whose underlying graph is A
is $\aleph_1^{\aleph_1} \cdot \aleph_1^{\aleph_1}$, and hence is a set.
simple.

then. Let \mathcal{F} be any full subcategory of \mathcal{F} - \mathcal{B} - \mathcal{C} . Let \mathcal{C} be a topological
 $A \subseteq K$ a subset of $C \times C$. Then there is a collection of pairs $(D_\alpha, \varphi_\alpha)$ indexed
set Q of \mathcal{F} - \mathcal{B} -morphisms φ_α , and such that:

(1) For each $\varphi_\alpha \in Q$, D_α is an object of \mathcal{F} , $\varphi_\alpha: C \rightarrow D_\alpha$ is a \mathcal{F} - \mathcal{B} -morphism generating
 \mathcal{D}_α and $\varphi_\alpha(a) = \varphi_\alpha(b)$ for all $a, b \in K$.

(2) If K is any object of \mathcal{F} and $k: C \rightarrow K$ is a \mathcal{F} - \mathcal{B} -morphism generating K , and
such that $k(a) = k(b)$ for all $a, b \in K$, then there is a φ_α in Q and isomorphism ψ :
 $D_\alpha \rightarrow K$ such that $\psi \varphi_\alpha = k$.

Proof. For each pair of cardinals \aleph_i, \aleph_j with $\aleph_i \leq \aleph_j \leq \max\{|C|, \aleph_1\}$, choose sets A
and B (Δ) such that $|A| = \aleph_i, |B| = \aleph_j$. Then there exists \mathcal{F} -graphs Γ with
set A of arrows set B (Δ) of objects. Let Q be the class of all pairs $(D_\alpha, \varphi_\alpha)$ where D_α
is a topological category with underlying graph such a Γ , and $\varphi_\alpha: C \rightarrow D_\alpha$ is a \mathcal{F} - \mathcal{B} -
morphism generating \mathcal{D}_α and such that $\varphi_\alpha(a) = \varphi_\alpha(b)$ whenever $a, b \in K$. Then Q
is a set, by Lemma 4.2, and condition (2) of the proposition is satisfied in consequence of Lemma 4.1.

Proof of Theorem 1. The proof now follows a standard pattern. This method
is similar to that used in proving Theorem 1 of [A 1] except that the proof there
avoids the set theory computations by working in a universe.

Let Q be the set given by Proposition 4.3. Then Q is non-empty (the trivial category with one object and one element is an object of \mathcal{F} , since \mathcal{F} is non-empty and admits arbitrary subobjects).

Let L be the product of the objects $D_q, q \in Q$, and let $g: C \rightarrow L$ be the $\mathcal{F}\mathcal{E}$ -morphism with components $q, q \in Q$. By our assumptions, L is an object of \mathcal{F} , as is the subcategory D of L generated by g . Let $p: C \rightarrow D$ be the restriction of g .

It is now a standard argument to show, using Proposition 4.3, that $p: C \rightarrow D$ presents D as the \mathcal{F} -category C with relations R .

Proof of Proposition 3. Let $i: C \rightarrow C^{op}$ be the identity antimorphism, and let $R^{op} = (i \times i)(R)$. It is easily verified that the two antimorphisms

$$\begin{aligned} C &\rightarrow C/\mathcal{F}R \rightarrow (C/\mathcal{F}R)^{op} \\ C &\rightarrow C^{op} \rightarrow C^{op}/\mathcal{F}R^{op} \end{aligned}$$

are both universal for anti-morphisms $C \rightarrow K$ which satisfy R , and so $C^{op}/\mathcal{F}R^{op} = (C/\mathcal{F}R)^{op}$.

Suppose now C is a topological groupoid. Let $\tilde{R} = \{(a^{-1}, b^{-1}); (a, b) \in R\}$. Then $f: C \rightarrow K$ satisfies the relations R if and only if f satisfies the relations \tilde{R} . Hence $C/\mathcal{F}R = C/\mathcal{F}\tilde{R}$.

Let $\kappa: C \rightarrow C^{op}$ be the isomorphism of topological groupoids $a \mapsto a^{-1}$. Then $(\kappa \times \kappa)(R) = \tilde{R}$, and so κ induces an isomorphism $\kappa^*: C/\mathcal{F}R \rightarrow C^{op}/\mathcal{F}\tilde{R}$. So we obtain an isomorphism, $\kappa^*: C/\mathcal{F}R \rightarrow (C/\mathcal{F}R)^{op}$. Clearly κ^* is $b \mapsto b^{-1}$. Therefore $C/\mathcal{F}R$ is a topological groupoid.

Proof of Corollary 4. Let \mathcal{F} be the full subcategory of $\mathcal{F}\mathcal{E}$ of all topological categories whose underlying category is an object of \mathcal{F} . Clearly \mathcal{F} admits arbitrary subobjects and products.

For any object C of \mathcal{E} , let $C' = \mathbf{L}_{\mathcal{E}}(C)$ (Example 8). Let R be any subset of $C \times C$. Then there is a morphism $p: C' \rightarrow C'/\mathcal{F}R$ making $C'/\mathcal{F}R$ the \mathcal{F} -category C' with relations R . But then it is easy to check that $\mathbf{F}_{\mathcal{E}}(p): C \rightarrow \mathbf{F}_{\mathcal{E}}(C'/\mathcal{F}R)$ is the \mathcal{F} -category C with relations R .

Proof of Proposition 7. Let G' be the abstract groupoid G/N with the identification topology (on arrows and on objects) given by $p: G \rightarrow G'$. It will be sufficient to prove that G' is a topological groupoid, i.e. that the structure functions are continuous.

The continuity of $\mathcal{E}, \beta: G' \rightarrow \text{Ob}(G'), \text{id}: \text{Ob}(G') \rightarrow G'$, and the inverse function $G' \rightarrow G'$ follows easily from the continuity of the corresponding maps for G and the fact that $G \rightarrow G', \text{Ob}(G) \rightarrow \text{Ob}(G')$ are identification maps. The only difficulty is proving continuity of composition.

Since $\text{Ob}(G)$ is HAUSDORFF, the domain $G \times G$ of the composition map θ in G is closed in $G \times G$.

We now prove $p: G \rightarrow G'$ is a proper map. By [2] p. 105, Proposition 9, we need only prove that the p -saturation of each compact set of G is compact. Let C

be a compact subset of G . Then $p^{-1}p(C) = NCN$. Now $NC = (p \tilde{\times} p)(N \tilde{\times} C)$. Since N is compact, so also is $N \times C$; since $N \tilde{\times} C$ is closed in $N \times C$, it is compact. Hence NC is compact. Similarly NCN is compact. So p is proper. Hence $p \times p: G \times G \rightarrow G' \times G'$ is a closed map. But $G \tilde{\times} G$ is closed in $G \times G$. Hence $p \tilde{\times} p: G \tilde{\times} G \rightarrow G' \tilde{\times} G'$ is a closed map, and hence an identification map. Continuity of composition in G' follows.

Proof of Proposition 8. Let X_0 be the subcategory of $X \times X$ with arrows (x, y) such that either $x = y$ or there is an arrow in C with domain and codomain objects in $\sigma^{-1}(x)$, $\sigma^{-1}(y)$ respectively. Let $j: X \rightarrow X_0$ be the inclusion of the pointlike category X and let $g: C \rightarrow X_0$ be $a \rightarrow (\sigma\delta'(a), \sigma\delta(a))$. Then $gi = ja$, and so there is a unique morphism $g^*: U_0(C) \rightarrow X_0$ such that $g^*g = g$, $g^*j = j$. The last condition implies that $\text{Ob}(g^*) \cap \text{Ob}(j) = 1_X$. But $\text{Ob}(j)$ is a bijection (this is known in the abstract case and so follows for the topological case). So $\text{Ob}(g^*)$ is a homeomorphism.

Proof of Proposition 9. Let $\sigma^*: X \times X \rightarrow U(X \times X)$ be the universal morphism, let $e \in X$, and let $k: X \rightarrow U(X \times X)$ be $x \rightarrow \sigma^*(e, x)$. Then $k(e)$ is the identity.

Suppose $g: X \rightarrow H$ is any continuous function to a topological group H such that $g(e)$ is the identity. We first prove uniqueness of a morphism $h: U(X \times X) \rightarrow H$ such that $hk = g$. Given $hk = g$, we find

$$h\sigma^*(x, y) = g(y)g(x)^{-1}$$

so that $h\sigma^*$, and hence h , is determined by g .

Conversely, if $h': U(X \times X) \rightarrow H$ is defined by $h'(x, y) = g(y)g(x)^{-1}$, then h' is a morphism to a topological group, and so defines $k: U(X \times X) \rightarrow H$ such that $h\sigma^* = k'$, whence $hk = g$.

This verifies the required universal property for k .

5. Problems

1. Find a topological groupoid G and totally disconnected normal subgroupoid N such that $p: G \rightarrow G/N$ is not an open mapping.
2. Find a topological groupoid G and normal subgroupoid N such that $p: G \rightarrow G/N$ is not an identification mapping.
3. Let $i: \mathcal{F}\mathcal{S} \rightarrow \mathcal{F}\mathcal{B}$ be the inclusion. Does i possess a right adjoint? Note that in the abstract case a right adjoint is constructed by assigning to any category C its subgroupoid of invertible elements.
4. Let G be a topological groupoid over X and $\sigma: X \rightarrow Y$ a continuous function. We know $\sigma^*: G \rightarrow U_0(G)$ is continuous, and is injective on the set G^0 of non-identity elements. Under what circumstances is $\sigma^*|_{G^0}$ a homeomorphism into? Notice that in the case $G = X \times G'$ and Y is a point, then a necessary condition for $\sigma^*|_{G^0}$ to be a homeomorphism into is that X be completely regular [16].

5. Graev has proved that the free product of HAUSDORFF topological groups is HAUSDORFF. This suggests the question: if G is a HAUSDORFF topological groupoid over X , $\sigma: X \rightarrow Y$ is continuous, and Y is HAUSDORFF, is then $U_\sigma(G)$ HAUSDORFF?
6. Relate the algebraic topology (e.g. homology groups or homotopy types) of $U_\sigma(G)$ and G . The only result that seems to be known on this problem is in the case $G = X \times X$, when $U(G) \cong FG(X)$; if X is a connected CW-complex, then $FG(X)$ is known to be of the homotopy type of the loops on the suspension of X [10].
7. It can be proved in a similar fashion to the abstract case (8.1.6 (Corollary 1) in [3]) that if G is a globally trivial topological groupoid over X then $U(G)$ is isomorphic to the topological free product $G\{x_0\} * FG(X)$. Find an example where this isomorphism is true abstractly but not topologically.

Note added in proof

Progress has been made on some of these problems.

2. An example with G finite has been given by Julie Berger.
5. It is proved in [A 3] that $U_\sigma(G)$ is Hausdorff if G is a HAUSDORFF k_σ -groupoid (see also [A 65]).
6. Hardy and Puppe have determined in [A 66] the homotopy type of the classifying space $BU_\sigma(G)$ under useful conditions on G . In particular, under their conditions, $BU_\sigma(G) \simeq BG/\text{Ob}(G)$.

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Appendix

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