Identities among relations

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Introduction

An "identity among relations" is for a presentation of a group what a "syzygy among relations", as considered by Hilbert, is for a presentation of a module. The notion has ramifications in topology as well as in combinatorial group theory, and in particular is involved in some difficult problems in the algebraic topology of 2-diminsional complexes. Our aim is an exposition of this area explaining the connections with the following topics:

This will give also some background to the "two dimensional group theory" of the article [Br2] in this volume, and will contain all the prerequisites for the (posthumous) article by P. Stefan in this volume [St].

1. Presentations and identities

We consider a presentation $P = (X; R)$ of a group G . Thus we have a short exact sequence

$$1 \longrightarrow N \longrightarrow F \longrightarrow G \longrightarrow 1$$

where F is the free group on the set X, R is a subset of F and $N = N(R)$ is the normal closure in F of the set R. The group F acts on N by conjugation $c \longmapsto c^u = u^{-1}cu$, for $c \in N$, $u \in F$, and the elements of N are of course all *consequences* of the set R, that is any $c \in N$ is of the form

$$c = (r_1^{\varepsilon_1})^{u_1} (r_2^{\varepsilon_2})^{u_2} \ldots (r_n^{\varepsilon_n})^{u_n}$$

where $r_i \in R$, $\varepsilon_i = \pm 1$, $u_i \in F$.

An *identity among relations* is, heuristically, such a specified product in which $c = 1$ in F. A formal definition is given below, but let us first consider some examples.

EXAMPLE 1. For any elements r, s of R we have the identities

$$r^{-1}s^{-1}rs^r = 1$$

$$rs^{-1}r^{-1}s^{\tilde{r}} = 1 \qquad \text{where } \tilde{r} = r^{-1} .$$

These identities hold always, whatever R.

EXAMPLE 2. Suppose $r \in R$, $s \in F$ and $r = s^m$. Then $rs = sr$ (i.e. s belongs to the centraliser $C(r)$ of r) and we have the identity

$$r^{-1}r^s = 1 .$$

However, for an element r of a free group F there is a unique element z of F such that $r = z^q$ with q maximal, and then $C(r)$ is the infinite cyclic group on z. This element z is called the *root* of r, and if $q > 1$, then r is called a *proper power*. So if $r^{-1}r^s = 1$, then s is a power z^n of the root of r (cf. [L-S], p.10, I.2.19).

EXAMPLE 3. Suppose the commutators $[x, y] = x^{-1}y^{-1}xy$, $[y, z]$, $[z, x]$ are among the relations. Then the well-known rule

$$[x, y][x, z]^y [y, z][y, x]^z [z, x][z, y]^x = 1$$

is an identity among the relations (since $[y, x] = [x, y]^{-1}$).

EXAMPLE 4. Consider the standard presentation $(x, y; r,s,t)$ of $\mathbb{Z}_2 \times \mathbb{Z}_2$ in which $r = x^2$, $s = y^2$, $t = x^{-1}y^{-1}xy$. We have an identity among relations

$$rts^{xy}(r^{-1})^y \, s^{-1}(t^{-1})^{\widetilde{\widetilde{yx}}} = 1 ,$$

as is easily checked. This identity may be read as a path starting at 1 in the Cayley diagram:

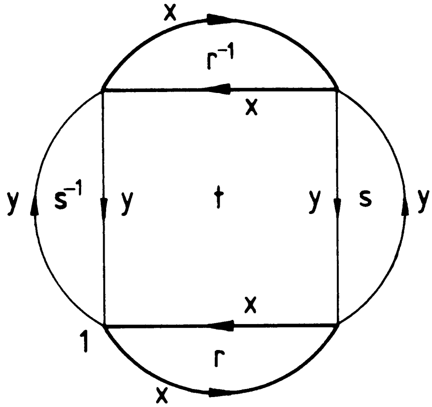


Figure 1

EXAMPLE 5. Consider the standard presentation $(x, y; r,s,t)$ of S_3 , the symmetric group on three letters, in which $r = x^3$, $s = y^2$, $t = xyxy$. We have an identity among relations

$$ts^{-1}t(s^{-1})^{x}(r^{-1})^{\tilde{y}x}t^{x}(s^{-1})^{xx}r^{-1} = 1$$

as is easily checked. This identity may be read as a path starting at 1 in the Cayley diagram:

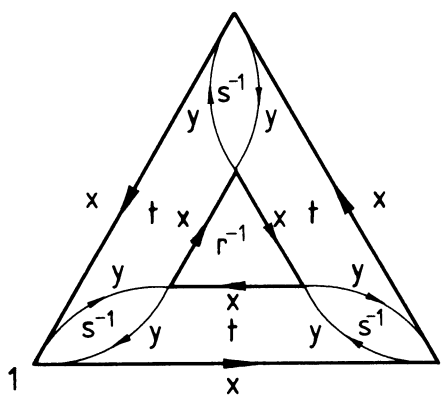


Figure 2

(Precise methods for obtaining the identities from these diagrams are given in §10).

Note that in these examples, conjugation is crucial. Indeed in the last example the elements r , s , t freely generate a subgroup of F .

The precise idea of *specifying* a consequence of the relations, and in particular of specifying an identity, is similar to that of specifying a relator as an element of a free group, but takes the action of F into account. The definitions are due to Peiffer and Reidemeister [Pe, Re2].

One extra formality is needed first. We wish to allow for repeated relations, and so regard a presentation $\mathcal{P}$ as a triple $(X; \mathcal{R}, w)$ where $\mathcal{R}$ is a set, $w: \mathcal{R} \longrightarrow F$ is a function to F , the free group on X , and $\mathcal{R}$ is assumed disjoint from F . The elements of $\mathcal{R}$ will be written $\rho, \sigma, \tau, \ldots$ and the elements $w\rho, w\sigma, w\tau, \ldots$ will be written $r, s, t \ldots$. We write $R = w(\mathcal{R})$ and $N = N(R)$ as above.

Let H be the free operator group on $\mathcal{R}$ with right operators $(h, u) \longmapsto h^u$, $h \in H, u \in F$, from F . Thus as a group H is free on the set $Y = \mathcal{R} \times F$, the elements of which are written ρ^u, $\rho \in \mathcal{R}$, $u \in F$, with ρ^1 written ρ and $(\rho^u)^{-1}$ written ρ^{-u} or $(\rho^{-1})^u$.

We mention an alternative way of obtaining H (cf. [Pe; Satz 3 on p.69], [Me2]).

PROPOSITION 1. *Let* $F(X \cup \mathcal{R})$ *denote the group freely generated by* $X \cup \mathcal{R}$. *Then* H *is isomorphic to the normal closure of* $\mathcal{R}$ *in* $F(X \cup \mathcal{R})$.

Proof. Clearly $F(X \cup \mathcal{R})$ contains F , the free group on X , and F is a Schreier transversal for the normal closure of $\mathcal{R}$ in $F(X \cup \mathcal{R})$. The Reidemeister-Schreier method gives the elements $u^{-1}\rho u$, $\rho \in \mathcal{R}$, $u \in F$, as a basis for this normal closure. □

Let $\theta: H \longrightarrow F$ be the homomorphism of groups given on the basis elements by

$$\theta(\rho^u) = u^{-1}ru , \quad \text{where} \quad r = w\rho,\ \rho \in \mathcal{R},\ u \in F .$$

If we let F act on itself by conjugation, then θ is an operator homomorphism, that is

$$\theta(h^u) = u^{-1}(\theta h)u,\ h \in H,\ u \in F .$$

Further $\theta(H) = N$.

We now define the *identities among the relations* for the presentation $P = (X; R, w)$ to be the elements of the kernel E of $\theta: H \longrightarrow F$.

However the group E contains certain identities which are always present, namely those corresponding to the identities in Example 1. We therefore consider in E the *basic Peiffer elements*, namely those of the form

$$p = a^{-1}b^{-1}ab^{\theta a}, \quad a, b \in Y .$$

More generally, any element of H of the form

$$p = h^{-1}k^{-1}hk^{\theta h}, \quad h, k \in H$$

will be called a *Peiffer element*; such an element is an identity, i.e. belongs to E . These elements were introduced in [Pe; p.70] (not in the form given above) and [Re2] .

The aim now is to factor out the Peiffer elements since these correspond to identities which are always present. It is convenient to discuss the situation in greater generality.

2. Pre-crossed and crossed modules

Let Γ be a group. A *pre-crossed* Γ*-module* (A, δ) consists of a group A ; a homomorphism $\delta: A \longrightarrow \Gamma$ of groups; and an action of Γ on the right of A , written $(a, u) \longmapsto a^u$, $a \in A$, $u \in \Gamma$. A sole condition imposed is:

CM1) $\delta(a^u) = u^{-1}(\delta a)u$, $a \in A$, $u \in \Gamma$.

If we regard Γ as acting on itself by conjugation, then CM1 says simply that δ is a Γ-morphism.

The pre-crossed Γ-module (A, δ) is a *crossed* Γ*-module* if it also satisfies

CM2) $a^{-1}ba = b^{\delta a}$, $a , b \in A$.

Let (A, δ) , (A', δ') be pre-crossed Γ-modules. A *morphism of pre-crossed* Γ*-modules* $\phi:(A, \delta) \longrightarrow (A', \delta')$ is a Γ-morphism $\phi: A \longrightarrow A'$ of groups such that $\delta'\phi = \delta$.

We shall construct from any pre-crossed Γ-module a crossed Γ-module.

Let (A, δ) be a pre-crossed Γ-module. We call the elements

$$\langle a, b \rangle = a^{-1}b^{-1}ab^{\delta a}$$

for all $a , b \in A$ the *Peiffer elements* of A . We call the subgroup of A generated by all Peiffer elements the *Peiffer group* of (A, δ). (We are here generalising a terminology used for the special case of the free pre-crossed F-module (H, θ) derived from a presentation. The Peiffer elements are then sometimes called *crossed commutators*; the elements of P are sometimes called *Peiffer identities*; and the term *Peiffer group* is used in [Me2].)

PROPOSITION 2. *Let* (A, δ) *be a pre-crossed* Γ*-module. Then the Peiffer group* P *of* (A, δ) *is normal in* A *and* Γ*-invariant.*

Proof. Let $a, b, c \in A$. Then

$$c^{-1} < a, b > c = < ac, b > < c, b^{\delta a} >^{-1} .$$

Thus a conjugate of a Peiffer element is a product of Peiffer elements, and so P is normal.

Let $a, b \in A$, $u \in \Gamma$. Then

$$< a, b >^{u} = < a^{u}, b^{u} >$$

(on using $\delta(a^{u}) = u^{-1}(\delta a)u$). So P is Γ-invariant. □

COROLLARY. *Let* (A, δ) *be a pre-crossed* Γ*-module. Then there is a crossed* Γ*-module* (C, ∂) *and a morphism* $\phi: (A, \delta) \longrightarrow (C, \partial)$ *of pre-crossed* Γ*-modules, such that* ϕ *is universal for morphisms from* (A, δ) *to crossed* Γ*-modules.*

Proof. Let P be the group defined above. Then the quotient group $C = A/P$ is well-defined, and C inherits a Γ-action and a Γ-morphism $\partial: C \longrightarrow \Gamma$. So (C, ∂) is a pre-crossed Γ-module. By definition of P, we have $c^{-1}dc = d^{\partial c}$ for all $c, d \in C$, and so (C, ∂) is a crossed Γ-module. The quotient morphism $A \longrightarrow C$ is clearly a morphism of pre-crossed Γ-modules and is universal for morphisms of (A, δ) to crossed Γ-modules. □

The above construction can be applied to the pre-crossed F-module (H, θ) constructed above from a presentation $\mathcal{P} = (X; R, w)$ of a group G. This will give a key example of a crossed F-module for F a free group.

In this example, we shall be interested in the way the Peiffer group of (H, θ) is generated. The general result for this is the following.

PROPOSITION 3. *Let* (A, δ) *be a pre-crossed* Γ*-module and let* V *be a set of generators for the group* A *such that* V *is* Γ*-invariant. Then the Peiffer group* P *of* (A, δ) *is the normal closure in* A *of the set* Z *of Peiffer elements* $< a, b >$

with $a, b \in V$.

Proof. Let P' be the normal closure in A of Z . Then $P' \subset P \subset \operatorname{Ker} \delta$. The rule $< a, b >^u = < a^u, b^u >$, $a, b \in A$, $u \in \Gamma$, shows that Z , and hence also P' , is Γ-invariant. So $C' = A/P'$ becomes a Γ-group and δ induces $\partial' : C' \longrightarrow \Gamma$ making (C', ∂') a pre-crossed Γ-module.

Since V generates A as a group, we have

$$x^{\partial' y} = y^{-1}xy \qquad (*)$$

for all x , y in a set V' which generates C' as a group. For fixed y , the set of x satisfying (*) is a subgroup of C' , so (*) is true for all $y \in V'$, $x \in C'$. Also the set of y satisfying (*) is closed under multiplication (because $x^{\partial'(yz)} = (x^{\partial' y})^{\partial' z} = z^{-1}(x^{\partial' y})\, z = z^{-1}y^{-1}xyz$) and under inversion (because if $x^{\partial' y^{-1}} = w$, then $x = w^{\partial' y} = y^{-1}wy$, so $w = yxy^{-1}$). It follows that (*) holds for all $x , y \in C'$, and hence $P \subset P'$. □

COROLLARY. *Let* (H, θ) *be the pre-crossed F-module derived from a presentation* $\mathcal{P} = (X; R, w)$. *Then the Peiffer group* P *of* (H, θ) *is the normal closure in* H *of the basic Peiffer elements* $< a, b >$, $a , b \in R \times F$. □

We have now constructed a useful family of crossed F-modules, namely, those derived from a presentation. These examples will be fundamental in later pages. Other examples of crossed Γ-modules are:

(i) (A, i) , in which i is the inclusion of a normal subgroup A of Γ and Γ acts on A by conjugation,

(ii) $(A, 0)$ in which A is a Γ-module in the usual sense and 0 is the constant map,

(iii) (A, δ) , where A is a group, $\Gamma = \operatorname{Aut} A$ acts on A in the obvious way, and $\delta: A \longrightarrow \operatorname{Aut} A$ assigns to a in A the inner automorphism $x \longmapsto a^{-1}xa$ of A .

Thus a crossed module generalises the concepts of a normal subgroup and that of an ordinary module.

We shall need some basic algebraic properties of crossed modules.

Let (A, ∂) be a crossed Γ-module. We write π for $\operatorname{Ker} \partial$ and N for $\operatorname{Im} \partial$. So we have an induced exact sequence of Γ-groups

$$1 \longrightarrow \pi \longrightarrow A \xrightarrow{\partial'} N \longrightarrow 1 \qquad (*).$$

We can now state some easy properties of these groups.

(2.1) N *is normal in* Γ *so that we can set* $G = \text{Coker}\ \partial$ *to obtain an exact sequence of groups*

$$1 \longrightarrow N \longrightarrow \Gamma \longrightarrow G \longrightarrow 1 .$$

(2.2) π *is contained in the centre* ZA *of* A *, and in particular* π *is abelian* .

(2.3) *The subgroup* N *of* Γ *acts trivially on* ZA *, and so also on* π *; hence* π *inherits an action of* $G = \Gamma/N$ *to become a* G*-module.*

(2.4) *The abelianised group* $\bar{A} = A/[A, A]$ *inherits a structure of* G*-module.*

This last result is proved by noting that $N = \partial A$ acts trivially on $\bar{A}$ since for $a, b \in A$ the element $(b^{-1})^{\partial a} b$ is the commutator $a^{-1}b^{-1}ab$, by (CM2) .

Since N is normal in Γ , the action of Γ on N by conjugation determines an action of G on the abelianised group $\bar{N} = N/[N, N]$, so that $\bar{N}$ becomes a G-module. It is clear that (*) determines an exact sequence of G-modules

$$\pi \longrightarrow \bar{A} \longrightarrow \bar{N} \longrightarrow 0 .$$

In general the map $\pi \longrightarrow \bar{A}$ is not injective. To see this, consider a group A and the crossed (Aut A)-module (A, δ) of (iii) above. Then $\pi = \text{Ker}\ \partial$ is the centre ZA of A . There are non-abelian groups A such that $1 \neq ZA \subset [A, A]$, for example the quaternion group, the dihedral groups D_{2n} , and many others. For all these, the composite $\pi = ZA \longrightarrow A \longrightarrow \bar{A}$ is trivial, and so not injective. This example gives point to the following result, which uses the notation of previous paragraphs.

PROPOSITION 4. *If the exact sequence* (*) *has a section (a group homomorphism but not necessarily a* Γ*-map), then* A *is isomorphic as group to* $\pi \times N$. *Further* $[A, A] \cap \pi = \{1\}$ *, the induced map* $\pi \longrightarrow \bar{A}$ *is injective, and so the sequence*

$$0 \longrightarrow \pi \longrightarrow \bar{A} \longrightarrow \bar{N} \longrightarrow 0$$

is a short exact sequence of Γ*-modules.*

Proof. Let $s\colon N \longrightarrow A$ be a section of (*) . Then $A = \pi \times sN$ (since π is in the centre of A). Since π is abelian, $[A, A] = [sN, sN]$ whence $[A, A] \cap \pi = \{1\}$. This implies $\pi \longrightarrow \bar{A}$ injective. □

These results apply to the crossed F-module (C, ∂) derived as in §1 from a presentation $P = (X; R, w)$. In this case F is a free group and hence so also is $N = \partial C$. The sequence $1 \longrightarrow \pi \longrightarrow C \xrightarrow{\partial'} N \longrightarrow 1$ has a section, and so Proposition 4 applies. The G-module $\bar{N}$ has been much studied - it is called the *relation module* of P (see [D1-4, G2, L1-2, We]) . We will see more of $\bar{N}$ later.

In the next section we present an important universal property of the crossed F-module (C, ∂) of a presentation P .

3. Free Crossed modules

Given a presentation $P = (X; R, w)$ we constructed in §1 a pre-crossed F-module (H, θ) , where F is the free group on X . From this we can construct a crossed F-module (C, ∂) . It is convenient to give these constructions in greater generality.

Let (A, δ) be a pre-crossed Γ-module, let R be a set and let $v: R \longrightarrow A$ be a function. We say (A, δ) is a *free pre-crossed* Γ*-module with basis* v if for any pre-crossed Γ-module (A', δ') and function $v': R \longrightarrow A'$ such that $\delta' v' = \delta v$, there is a unique morphism $\phi: (A, \delta) \longrightarrow (A', \delta')$ of pre-crossed Γ-modules such that $\phi v = v'$. In such case, we also emphasise the rôle of the function $w = \delta v: R \longrightarrow \Gamma$ by calling (A, δ) , with the function v , a *free pre-crossed* Γ*-module on* w . If (A, δ) is a crossed Γ-module, and (A, δ) with v has the above universal property for maps into crossed Γ-modules, then we call (A, δ) a *free crossed* Γ*-module with basis* v (or *on* w) .

PROPOSITION 5. *Let* Γ *be a group,* R *a set and* $w: R \longrightarrow \Gamma$ *a function. Then a free pre-crossed* Γ*-module on* w *, and a free crossed* Γ*-module on* w *, exist, and are each uniquely determined up to isomorphism.*

Proof. For the existence of a free pre-crossed Γ-module we generalise easily the construction of §1 . That is, we let H be the free group on the set $R \times \Gamma$, and write the elements of this set as ρ^u, $\rho \in R$, $u \in \Gamma$. Let Γ act on H by acting on the generators as $(\rho^u)^v = \rho^{uv}$, $\rho \in R$, u , $v \in \Gamma$. Define $\theta: H \to \Gamma$ by its values on the generators

$$\theta(\rho^u) = u^{-1}(w\rho)u \text{ , } \rho \in R \text{ , } u \in \Gamma \text{ .}$$

Define $v: R \longrightarrow H$ by $v(\rho) = \rho^1$, $\rho \in R$, so that $\delta v = w$. Then (H, θ) is a pre-crossed Γ-module, which, with v , is easily checked to be a free pre-crossed Γ-module on w .

From (H, θ) we can form a crossed Γ-module (C, ∂) by factoring out the Peiffer elements. Then (C, ∂) with the composite $R \longrightarrow H \longrightarrow C$ is a free crossed Γ-module on w.

The uniqueness of these constructions up to isomorphism follows by the usual universal argument. □

The definition and construction of a free crossed Γ-module is due to Whitehead [Wh3]. We have followed a suggestion of P.J. Higgins and shown the rôle of the pre-crossed modules, since they are used implicitly also in some later proofs.

It is clear from the construction of the free pre-crossed Γ-module (H, θ) on $w : R \longrightarrow \Gamma$ that the basis function $v : R \longrightarrow H$ is injective. We wish to have this result for the free crossed Γ-module.

PROPOSITION 6. *Let* (C, ∂) *be the free crossed* Γ*-module on* $w : R \longrightarrow \Gamma$, *with basis function* $v : R \longrightarrow \Gamma$. *Then* v *is injective.*

This is most easily proved using the following result.

PROPOSITION 7. *If* (C, ∂) *is a free crossed* Γ*-module, with basis* $v : R \longrightarrow C$, *and* $G = \text{Coker } \partial$, *then the abelianised group* $\overline{C}$ *is a G-module that is free on the composition* $\overline{v} : R \xrightarrow{v} C \longrightarrow \overline{C}$. *Hence* $\overline{v}$, *and so also* v, *is injective.*

Proof. By (2.4), $\overline{C}$ has the structure of G-module.

Let $p : \Gamma \longrightarrow G$ be the quotient map, and let M be a G-module. Then $\Gamma \times M$, with projection to Γ, becomes a crossed Γ-module with action of Γ by conjugation on Γ and *via* p on M.

Let $v' : R \longrightarrow M$ be a function. Define $v'' = (\partial v, v') : R \longrightarrow \Gamma \times M$. Freeness of C gives a morphism $\phi : C \longrightarrow \Gamma \times M$ of crossed Γ-modules such that $\phi v = v''$. Composition with projection gives $\phi' : C \longrightarrow M$, a morphism of groups which factors through $\overline{\phi} : \overline{C} \longrightarrow M$. This is a G-morphism as required. □

We can now regard v and $\overline{v}$ as inclusions, and link Proposition 7 nicely with Proposition 4.

COROLLARY. *Let* (C, ∂) *be the free crossed* F*-module constructed as above from a presentation* $(X; R, w)$ *of a group* G. *Write* $\pi = \text{Ker } \partial$, $N = \text{Im } \partial$. *Then the induced map* $j : \pi \longrightarrow \overline{C}$ *is injective and there is an exact sequence of* G*-modules*

$$0 \longrightarrow \pi \xrightarrow{j} \overline{C} \xrightarrow{d} N \longrightarrow 0$$

in which $\bar{C}$ *is the free* G-*module on the elements* $\bar{v}(\rho) = \rho[C, C]$, $\rho \in R$, *and* d *is given by* $d(\rho[C, C]) = w(\rho)[N, N]$, $\rho \in R$.

Proof. Since F is free, so also is N , and so the surjection $\partial' : C \longrightarrow N$ has a section. □

From now on, we call π the *module of identities* for $P = (X; R, w)$, or the *module of identities among the relations of* P .

4. The Associated chain complex

Given a presentation $P = (X; R, w)$ of a group G , there is a standard way of constructing a chain complex of free (right) G-modules

$$C(P): C_2(P) \xrightarrow{d_2} C_1(P) \xrightarrow{d_1} C_0(P)$$

in which

$$C_0(P) = \mathbb{Z}G ,$$

$$C_1(P) = \bigoplus_X \mathbb{Z}G ,$$

$$C_2(P) = \bigoplus_R \mathbb{Z}G$$

with bases (as $\mathbb{Z}G$-modules) respectively 1; e^1_x for $x \in X$; and e^2_ρ for $\rho \in R$. Let F be the free group on X and let the projections $F \longrightarrow G$, $\mathbb{Z}F \longrightarrow \mathbb{Z}G$ determined by the presentation both be denoted by ϕ . Then the boundaries are given by

$$d_1(e^1_x) = 1 - \phi x , \quad x \in X$$

$$d_2(e^2_\rho) = \sum_X e^1_x . \phi(\partial r/\partial x) , \quad \rho \in R$$

where $\partial r/\partial x$ is the element of $\mathbb{Z}F$ known as the *Reidemeister-Fox derivative* of $r = w(\rho)$. It is computed as follows (see for example [C-F], [Bi]) .

First recall that a *derivation* f from a group Γ to a (right) Γ-module M is a function $f: \Gamma \rightarrow M$ satisfying

$$f(uv) = f(u).v + f(v), \quad u , v \in \Gamma \qquad (4.1).$$

This implies that $f(1) = 0$ and that

$$f(u^{-1}) = -f(u).u^{-1}, \quad u \in \Gamma \qquad (4.2).$$

From (4.1) it follows that if $u \in \Gamma$ can be written as $u = y_1 \dots y_n$ then

$$f(u) = f(y_1)\, y_2 \dots y_n + f(y_2)\, y_3 \dots y_n + \dots + f(y_n) \qquad (4.3).$$

It follows from this and (4.2) that if $\Gamma = F$, the free group on X, then a derivation f on F may be computed from the values $f(x)$, $x \in X$. It is also not hard to prove the converse, that given the values $f(x)$, $x \in X$, the formulae (4.2) and (4.3)(with $y_i \in X \cup X^{-1}$) determine uniquely a derivation f. (The neatest proof relies on the fact that the derivations $F \longrightarrow M$ are bijective with the right inverses of the projection $M \tilde{\times} F \longrightarrow F$ of the semi-direct product of M and F, cf. for example [Hi-St], p.196.)

It follows that for any x in X there is a unique derivation $F \longrightarrow \mathbb{Z}F$ whose value on a basis element $x' \in X$ is $\delta_{xx'}$ (the Kronecker delta). This derivation is written $\partial/\partial x$.
We can now give the formula for d_2 as follows: suppose $w(\rho) = y_1 \dots y_n \in F$ where $y_i \in X \cup X^{-1}$, $i = 1, \dots, n$; then

$$d_2(e^2_\rho) = \sum_{i=1}^{n} \hat{y}_i \cdot \phi(y_{i+1}\, y_{i+2} \dots y_n) \qquad (4.4)$$

where if $y \in X \cup X^{-1}$,

$$\hat{y} = \begin{cases} e^1_x & \text{if } y = x \in X, \\ -e^1_x(\phi x)^{-1} & \text{if } y^{-1} = x \in X. \end{cases}$$

We shall use this formula for d_2 in §5.

There is another way of expressing this formula. For any group Γ the functor $\mathrm{Der}(\Gamma, -)$ (of derivations from Γ to $-$) is represented by the augmentation ideal $I\Gamma$, and any derivation $f\colon \Gamma \longrightarrow M$ is uniquely the composite of a homomorphism $f^*\colon I\Gamma \longrightarrow M$ of Γ-modules and the derivation $\Gamma \longrightarrow I\Gamma$, $u \longmapsto 1 - u$ (see for example [Hi-St], p.194). If $\Gamma = F$ as above, then $I\Gamma$ is the free Γ-module on the elements $1 - x, x \in X$ (*loc. cit.* p.196), and so one may identify $\oplus_X \mathbb{Z}F$ and IF by the rule $e^1_x \longmapsto 1 - x$. So one has an identification

$$\oplus_X \mathbb{Z}G \longrightarrow IF \otimes_F \mathbb{Z}G$$

given by $e^1_x \longmapsto (1 - x) \otimes 1$. With this identification d_2 may be described simply as

$$d_2(e^2_\rho) = (1 - r) \otimes 1 \in IF \otimes_F \mathbb{Z}G \ , \quad \text{where} \quad r = w(\rho).$$

This description is often convenient in homological algebra.

PROPOSITION 8. *The module* π *of identities for* $P = (X; R, w)$ *is isomorphic to the second homology module* $H_2(C(P))$ *, i.e. to the kernel of* d_2 *.*

Proof. In the previous section we have constructed an exact sequence of G-modules

$$0 \longrightarrow \pi \longrightarrow \bar{C} \xrightarrow{d} \bar{N} \longrightarrow 0 \ .$$

We shall prove later (Corollary 1 to Proposition 9) that the rule $r \longmapsto d_2(e^2_\rho)$ $(r = w\rho)$ induces an injection $i: \bar{N} \longrightarrow \bigoplus_X \mathbb{Z}G$; an algebraic proof of this, using the latter form of d_2 , is given for example on p.199 of [Hi-St]. So we have a commutative diagram

$$\begin{array}{ccc} \bar{C} & \xrightarrow{\cong} & C_2(P) \\ {\scriptstyle d}\downarrow & & \downarrow{\scriptstyle d_2} \\ \bar{N} & \xrightarrow{i} & C_1(P) \end{array}$$

with i injective, and hence π is isomorphic to the kernel of d_2 . □

5. Relation with 2-dimensional CW-complexes

Let K be a connected CW-complex of dimension 2. Shrinking a tree in K^1 to a point does not change the homotopy type of K, and so we assume that K has only one vertex, say a . Then the fundamental group $G = \pi_1(K, a)$ has a presentation $(X; R, w)$ such that the elements x of X are bijective with the 1-cells e^1_x of K ; the elements ρ of R are bijective with the 2-cells e^2_ρ of K ; and the relators $r = w(\rho)$, $\rho \in R$ are determined up to conjugacy by the attaching maps $f_\rho : S^1 \longrightarrow K^1$ of the e^2_ρ's .

Conversely, given a presentation $P = (X; R, w)$ of a group G , one can form a CW-complex $K = K(P)$ with one vertex a ; a 1-cell e^1_x for each element x of X (so that $\pi_1(K^1, a)$ is the free group F on X) , and a 2-cell e^2_ρ for each element ρ of R , attached by a representative of the relator $r = w\rho$ in F . The homotopy type of $K(P)$ (and in fact the simple homotopy type [S1, Wr]) is independent of the choice of representative attaching maps for the 2-cells. Note also that for $K(P)$, the attaching maps f_ρ preserve the base point, i.e. $f_\rho(1) = a$. We call $K(P)$ the *geometric realisation* of P.

We shall show how to identify the chain complex $C(P)$ of the presentation with the cellular chain complex of the universal cover $\tilde{K}$ of $K = K(P)$. For this, recall that the cells of $\tilde{K}$ have characteristic maps which are precisely the lifts of the characteristic maps of the cells of K. This gives a convenient notation for the cells of $\tilde{K}$ as follows.

The set $\tilde{K}^0$ of vertices of $\tilde{K}$ is simply $G = \pi_1(K, a)$. The 1-cells of $\tilde{K}$ are bijective with $X \times G$ and so are written $e^1_{(x, g)}$, $(x, g) \in X \times G$, and $e^1_{(x, g)}$ joins $(\phi x)g$ to g, where $\phi\colon F \longrightarrow G$ is the projection. We write the edge path along $e^1_{(x, g)}$ as (x, g) and its inverse as $(x, g)^{-1} = (x^{-1}, (\phi x)\, g)$. The 2-cells of $\tilde{K}$ are bijective with $R \times G$ and so are written as $e^2_{(\rho, g)}$, $(\rho, g) \in R \times G$, this cell being attached by a map $f^g_\rho\colon S^1 \longrightarrow \tilde{K}^1$ lifting the attaching map f_ρ of e^2_ρ. Suppose the class of f_ρ in $F = \pi_1(K^1, a)$ is $r = y_1 \dots y_n$ where $y_i \in X \cup X^{-1}$, $i = 1, \dots, n$. Then by the uniqueness of path-lifting, the class of f^g_ρ in $\pi_1(\tilde{K}_1, g)$ contains the edge path

$$\left.\begin{array}{l} \qquad\qquad (y_1, g_1)\ (y_2, g_2)\ \dots\ (y_n, g_n) \\ \text{where} \\ g_1 = \phi(y_2 \dots y_n)\, g,\ g_2 = \phi(y_3, \dots, y_n)\, g, \dots, g_n = g\ . \end{array}\right\} \quad (5.1)$$

PROPOSITION 9. *The cellular chain complex* $(C_*(\tilde{K}), \partial)$ *of the universal cover* $\tilde{K}$ *of* $K = K(P)$ *is G-isomorphic to the chain complex* $C(P)$ *associated to the presentation* P.

Proof. The cellular chain group $C_i(\tilde{K}) = H_i(\tilde{K}^i, \tilde{K}^{i-1})$ is the free abelian group on the i-cells of $\tilde{K}$, and so has a base which can be identified with $R \times G$ if $i = 2$, with $X \times G$ if $i = 1$ and with G if $i = 0$. Since $e^1_{(x, g)}$ joins $(\phi x)g$ to g we have

$$\partial_1(x, g) = (1 - \phi x)g\ , \quad (x, g) \in X \times G\ .$$

Suppose $(\rho, g) \in R \times G$ and $w\rho = y_1 \dots y_n$ where $y_i \in X \cup X^{-1}$. Then by the description above of f^g_ρ

$$\partial_2(\rho, g) = (y_1, g_1) + (y_2, g_2) + \dots + (y_n, g_n)$$

where the g_i are given by (5.1). It follows from this and (4.4) that the map $C_*(\tilde{K}) \longrightarrow C(P)$ given on the basis elements by $(\rho, g) \longmapsto e^2_\rho \cdot g$ in dimension 2 ; $(x, g) \longmapsto e^1_x \cdot g$ in dimension 1 , and $g \longmapsto g$ in dimension 0 , is an isomorphism. □

COROLLARY 1. *If* $P = (X; R, w)$ *is a presentation of a group* G *and* $\bar{N}$ *is the relation module of* P *, then the rule* $r \longmapsto \Sigma e^1_x \, \phi(\partial r/\partial x)$ *induces an injection* $i: \bar{N} \longrightarrow \bigoplus_X \mathbb{Z}G$.

Proof. We have identifications

$$\bigoplus_X \mathbb{Z}G = C_1(\tilde{K}) = H_1(\tilde{K}^1, \tilde{K}^0) .$$

The homology exact sequence of the pair $(\tilde{K}^1, \tilde{K}^0)$ gives an injection $j: H_1(\tilde{K}^1) \longrightarrow H_1(\tilde{K}^1, \tilde{K}^0)$. The covering projection $p: \tilde{K} \longrightarrow K$ induces an isomorphism $\pi_1(\tilde{K}^1, 1) \longrightarrow N$ which maps the class of the edge path $(y_1, g_1) \dots (y_n, g_n)$ (as in (5.1)) to $r = y_1 \dots y_n$, $y_i \in X \cup X^{-1}$. The Hurewicz map $\pi_1(\tilde{K}^1, 1) \longrightarrow H_1(\tilde{K}^1)$ thus induces an isomorphism $\bar{N} \longrightarrow H_1(\tilde{K}^1)$ and the composite of j with this isomorphism is the map i . □

Corollary 1 may also be proved using the methods of §3.1 of [G1] . Given a group G , and short exact sequence $1 \longrightarrow N \longrightarrow F \longrightarrow G \longrightarrow 1$ with F free, Gruenberg constructs a free G-resolution of $\mathbb{Z}$

$$\dots \to \mathsf{N}^2/\mathsf{N}^3 \to \mathsf{FN}/\mathsf{FN}^2 \to \mathsf{N}/\mathsf{N}^2 \to \mathsf{F}/\mathsf{FN} \to \mathbb{Z}G \to \mathbb{Z} \to 0$$

in which F is the augmentation ideal IF of F and N is the kernel of the induced map $\mathbb{Z}F \longrightarrow \mathbb{Z}G$. If F is free on X , and N is the free group on V , then N/N^2 and F/FN are free G-modules on the cosets of the elements $1 - v$, $v \in V$ and $1 - x$, $x \in X$, respectively. Thus F/FN is isomorphic to our $C_1(P)$ but in general N/N^2 is not isomorphic to $C_2(P)$. However the map $N \longrightarrow \mathsf{N} \longrightarrow \mathsf{N}/\mathsf{FN}$ (which sends $n \in N$ to the coset of $1 - n$) induces an isomorphism of abelian groups $\bar{N} \longrightarrow \mathsf{N}/\mathsf{FN}$, and the map $\mathsf{N}/\mathsf{FN} \longrightarrow \mathsf{F}/\mathsf{FN}$ is an injection.

COROLLARY 2. *Let* $P = (X; R, w)$ *be a presentation of a group* G *, and let* $K = K(P)$ *be its geometric realisation. Then the module* π *of identities for* P *is naturally isomorphic to the second homology group* $H_2(\tilde{K})$ *of the universal cover* $\tilde{K}$ *of* K *,* *and hence also to* $\pi_2(K)$ *, the second homotopy group of* K .

Proof. The first assertion is immediate from Propositions 7 and 8, while the second follows from the Hurewicz theorem, since $\pi_2(\tilde{K}) \cong \pi_2(K)$. □

The above description of the module of identities as an absolute homotopy group can be extended to a description of the free crossed F-module C of the presentation as a relative homotopy group. The history of this description is as follows.

In his 1941 paper [Wh1] , Whitehead attempted an algebraic description of the second homotopy group $\pi_2(K)$ of a space $K = L \cup \{e^2_\rho\}_{\rho\in R}$ obtained by attaching 2-cells e^2_ρ to a path-connected space L . He reformulated these results in [Wh2] as a precise algebraic description of the group $\pi_2(K, L)$ and also noted that if L is a 1-dimensional complex, then his description of $\pi_2(K)$ returned to previous results of Reidemeister [Re1] (see Corollary 2 above).

A fundamental observation in [Wh2] is that if (Z, Y) is any based pair of spaces, then the second relative homotopy group $\pi_2(Z, Y)$ has an action of $\pi_1(Y)$ so that with the boundary map $\partial: \pi_2(Z, Y) \longrightarrow \pi_1(Y)$, the rules (CM1), (CM2) hold. (For proofs of these rules, see for example [Hi] p.39 or [W] .) This led Whitehead to the definition of crossed module [Wh3] .

For the particular pair (K, L) , where $K = L \cup \{e^2_\rho\}_{\rho\in R}$ as above, we can obtain elements $a_\rho \in \pi_2(K, L)$, given the characteristic maps $h_\rho: (E^2, S^1) \longrightarrow (K, L)$ of the 2-cells e^2_ρ together with a choice of paths in L , one for each ρ , joining $h_\rho(1)$ to the base point of L . We can now state a theorem from [Wh3] .

THEOREM 10. *The crossed* $\pi_1 L$*-module* $\pi_2(K, L)$ *, where* $K = L \cup \{e^2_\rho\}_{\rho\in R}$ *, is free on the elements* a_ρ , $\rho \in R$.

COROLLARY. *If* $K = K(P)$ *is the geometric realisation of a presentation* P *, then the free crossed* F*-module* (C, ∂) *of* P *is isomorphic, given the identification* $F = \pi_1 K^1$ *, to the crossed* $\pi_1 K^1$*-module* $(\pi_2(K^2, K^1), \partial)$*; in particular, the module* π *of identities of* P *is isomorphic to* $\pi_2(K)$. □

Whitehead's proof of Theorem 10 uses methods of transversality and knot theory - an exposition of this proof is given in [Br1] . The theorem is also a special case of the generalised Seifert-van Kampen theorem of [B-H2], sketched in [Br2] in this volume. In the case L is a 1-dimensional CW-complex, a short proof was given in [C1] as an application of the relative Hurewicz theorem, and this method has been extended to the general case in [R] . For completeness, we give another proof here of the special case, without using the Hurewicz theorem.

Proof of Theorem 10 *for the case* $L = K^1$. Clearly we may assume K is of the form $K(P)$ for a presentation $P = (X; R, w)$ of a group G . Let (C, ∂) be the free crossed F-module on $w: R \to F$, as in §3 . Then there is a unique homomorphism $\phi: C \longrightarrow \pi_2(K^2, K^1)$ of crossed F-modules such that $\phi\rho = a_\rho$, $\rho \in R$. So we obtain a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \pi & \xrightarrow{\;i\;} & C & \longrightarrow & N & \longrightarrow & 1 \\ & & \big\downarrow{\scriptstyle \phi'} & & \big\downarrow{\scriptstyle \phi} & & \big\| & & \\ 0 & \longrightarrow & \pi_2 K & \xrightarrow[j]{} & \pi_2(K^2, K^1) & \longrightarrow & N & \longrightarrow & 1 \end{array} .$$

That ϕ is surjective is fairly easily proved by a general position argument which will be given in §10 below (cf. also [W] and [Br1]) . The more difficult part is to prove ϕ injective.

We have isomorphisms given previously

$$\overline{C} \cong \bigoplus_R \mathbb{Z}G \cong H_2(\tilde{K}, \tilde{K}^1) .$$

Thus the Hurewicz map $\pi_2(K, K^1) \longrightarrow H_2(\tilde{K}, \tilde{K}^1)$ determines a map $\psi: \pi_2(K, K^1) \longrightarrow \overline{C}$ such that $\psi(a_\rho) = e^2_\rho$, $\rho \in R$. Hence, the abelianised maps $\overline{\psi}$, $\overline{\phi}$ satisfy $\overline{\psi}\overline{\phi} = 1$. Let $q: C \longrightarrow \overline{C}$, $q': \pi_2(K^2, K^1) \longrightarrow \overline{\pi}_2(K^2, K^1)$ be the abelianising maps. Then

$$\overline{\psi}\, q' \phi i = \overline{\psi}\, \overline{\phi}\, q\, i = q\, i$$

which is injective by Proposition 4 . Hence ϕi is injective and so ϕ' is injective. By the 5-lemma applied to the above diagram, ϕ is injective. □

REMARKS 1. These results do give precise information on $\pi_2(K)$, where K is a 2-complex, or equivalently, on the module of identities for a presentation $(X; R, w)$; however they are not so easy to interpret in practice. Quite a lot of information is known on relation modules, particularly for abelian groups [G2, S-D, We] . See [D4, Hu2, G-R] for some results on $\pi_2(K)$.

2. Dyer-Vasquez [D-V] have a different method of constructing a complex $K(P)$ of a one-relator presentation $P = (X; r)$ of a group G. If r is not a proper power, they proceed as above. However if $r = z^q$, where $q > 1$ is maximal, then they attach to K^1 not a 2-cell but an Eilenberg-MacLane space $K(Z_q, 1) = S^1 \cup_q e^2 \cup e^3 \cup \ldots$ by means of a map $S^1 \longrightarrow K^1$ representing z. This yields an Eilenberg-MacLane space $K(G, 1)$.

3. If P, P' are two presentations of a group G, then (see for example [C-F], [J]) P can be transformed to P' by a sequence of Tietze transformations, which are

I (and I') Add (delete) a generator and relation which expresses that generator as a word in the other generators, e.g.

$$(x, y;\ x^2 = y^3) \longmapsto (x, y, z;\ x^2 = y^3,\ z = x^{-1}y^2x)$$

II (and II') Add (delete) a relation which is a consequence of the other relations e.g.

$$(x, y;\ x^2 = y^3) \longmapsto (x, y;\ x^2 = y^3,\ x^4 = y^6).$$

Instead of the transformations II and II' one can also use the transformations (cf. [D4, Me1, S1, Wa]):

II a Replace a relator r by $rw^{-1}sw$ or $rw^{-1}s^{-1}w$ where s is another relator and w is an arbitrary word in the generators.

II b (and II b') Add (delete) the relation "1 = 1" (the corresponding relator is the identity).

A transformation II a is a product of a transformation II and a transformation II'. A transformation II (II') can easily be written as a product of transformations II a and II b(II b').

Sieradski in [S1] calls presentations equivalent under the use of operations I, I' and II a *combinatorially equivalent*. (Actually, he used a different, but equivalent, set of operations.) There are a number of problems in this area. The following is taken from Problem 5.1 of [K].

Let $K(P)$, $K(P')$ be the geometric realisations of two finite presentations P, P' of a group G. Assume P, P' have the same *deficiency* (= number of generators - number of relators). Consider the assertions:

A) $K(P) \simeq K(P')$ (homotopy equivalence)

B) $K(P) \curlywedge K(P')$ (simple homotopy equivalence)

C) $K(P) \underset{3}{\curlywedge} K(P')$ (simple homotopy equivalence by moves of dimension ≤ 3)

D) P is combinatorially equivalent to P' .

Then $D \Leftrightarrow C \Rightarrow B \Rightarrow A$ [Wr] (see also [Me 1, 2]) . It is not known what other relations hold in general. For more discussion of this and other problems on 2-complexes, see also [Wa2] .

The main results of [S1] and [Me1] give presentations of finitely generated abelian groups which are not combinatorially equivalent. The simplest example is the two presentations $(x, y; x^5, y^5, [x, y])$ and $(x, y; x^5, y^5, [x^2, y])$ of $\mathbb{Z}_5 \times \mathbb{Z}_5$. The proof involved considering the map $C(P) \longrightarrow C(P')$ of chain complexes induced by a combinatorial equivalence from P to P' . [Me1] also considers coarser equivalences (allowing also permutations of the generators).

The presentation $P = (x, y; x^2y^{-3}, 1)$ of the trefoil group G has its module π of identities isomorphic to $\mathbb{Z}G$. In [D4], Dunwoody constructs for G another presentation P' , with two generators and two relators, for which the module π' of identities is not free. Since π and π' are not isomorphic, the spaces K(P), K(P') are not of the same homotopy type. However, he also proves that $\pi \oplus \mathbb{Z}G \cong \pi' \oplus \mathbb{Z}G$, and that $K(P) \vee S^2$ and $K(P') \vee S^2$ are of the same homotopy type. Thus π' is projective and stably free. Other examples of non-free projective modules over $\mathbb{Z}G$ where G is torsion free are given in [Be-Du] (for G the trefoil group made matabelian), but it is not known if these are isomorphic to the second homotopy module of a 2-complex. Also by the Corollary on p.139 of [Wald] , $\tilde{K}_0(\mathbb{Z}G) = 0$ for G in a large class which includes (by Theorem 17.5 *op. cit.*) all poly-$\mathbb{Z}$-groups, all torsion free one-relator groups, and fundamental groups of compact, orientable 3-manifolds which are sufficiently large (this includes for example the trefoil group). Hence projective modules over such groups are stably free. Many examples of non-free projective modules over the rational group ring of a torsion free group are known [Le] , but the existence of these does not imply such examples exist over the integral group ring. For further discussion of related problems, see [Ba] .

6. Peiffer transformations

As before, let P = (X; R, w) be a presentation of a group G . Let F be the free group on X . The aim of this section is to give a more combinatorial description of the free crossed F-module C on $w : R \longrightarrow F$. This description, which is essentially due to Peiffer [Pe], will be useful later.

Recall that we considered in §1 the free group H on the set

$Y = R \times F$, with elements of Y written $a = \rho^u$, $\rho \in R$, $u \in F$. The combinatorial description of C uses operations on words rather than on elements of H ; a word in the elements of Y is written as an n-tuple

$$\underset{\sim}{y} = (a_1, \ldots, a_n), \text{ where } a_i = (\rho_i^{u_i})^{\varepsilon_i}, \varepsilon_i = \pm 1, \rho_i \in R, u_i \in F,$$

for some $n \geq 0$. We shall refer to such a sequence as a Y-*sequence*, and shall write $\theta\underset{\sim}{y}$ for the product $(\theta a_1) \ldots (\theta a_n)$ in F , where, as in §1, $\theta((\rho^u)^\varepsilon) = u^{-1}(w\rho)^\varepsilon u$. If $\theta y = 1$, we call $\underset{\sim}{y}$ an *identity* Y-*sequence* for P . For example,

$$\underset{\sim}{y} = (\rho^u, \rho^{-u}, \sigma^{-v}, \sigma^v, \tau^w, \tau^{-w})$$

where $\sigma, \rho, \tau \in R$ and $u, v, w \in F$, is an identity Y-sequence, and in this case the corresponding element of H is 1 .

In §2 we formed the Peiffer group P of the pre-crossed F-module (H, θ) as the subgroup of H generated by the Peiffer elements $b^{-1}a^{-1}ba^{\theta b}$ for all $a , b \in H$. This means that if we work mod P in H we have the "crossed commutation" rules

$$ab \equiv ba^{\theta b}, \quad ab \equiv b^{\theta a^{-1}} a \bmod P$$

for all $a, b \in H$. Further, the Corollary to Proposition 3 shows that these rules for all a , b of H are a consequence of the rules simply for all elements a , b of Y . Such rules, together with the rule $a\,a^{-1} = 1$, can be modelled on words in Y , by certain operations which we now explain.

Peiffer operations on Y-sequences:

(i) An *elementary Peiffer exchange* replaces an adjacent pair (a, b) in a Y-sequence by either $(b, a^{\theta b})$ or $(b^{\theta a^{-1}}, a)$. A *Peiffer exchange* is a sequence of elementary Peiffer exchanges; we often abbreviate "Peiffer exchange" to "exchange".

(ii) A *Peiffer deletion* deletes an adjacent pair (a, a^{-1}) in a Y-sequence. A *Peiffer collapse* is a sequence of exchanges and Peiffer deletions, in some order.

(iii) A *Peiffer insertion* is the inverse of a Peiffer deletion, and a *Peiffer expansion* is the inverse of a Peiffer collapse.

(iv) A *Peiffer equivalence* is a sequence of Peiffer collapses and Peiffer expansions, in some order.

REMARK. Operations of this kind are considered in [Pe] . A number of authors have used some coarser operations which we shall discuss later and call simply *collapses*, *expansions* and *equivalences*.

Given a Y-sequence $\underset{\sim}{y}$, we obtain an element $\psi\underset{\sim}{y}$ of the free group H on Y by forming the product in H of the components of $\underset{\sim}{y}$. By the construction of free groups, $\psi\underset{\sim}{y} = \psi\underset{\sim}{z}$ if and only if $\underset{\sim}{z}$ can be obtained from $\underset{\sim}{y}$ by a sequence of Peiffer deletions and Peiffer insertions, in some order. The definitions of the Peiffer group P and the free crossed F-module $C = H/P$ give immediately:

PROPOSITION 11. *Two* Y*-sequences have the same image in* $C = H/P$ *if and only if they are Peiffer equivalent.*

For later use, we also give a simple but useful observation on exchange operations.

PROPOSITION 12. *If a* Y*-sequence* $\underset{\sim}{z}$ *is obtained from a* Y*-sequence* $\underset{\sim}{y} = (a_1, \ldots, a_m)$ *by Peiffer exchanges, then each component of* $\underset{\sim}{z}$ *is of the form*

$$a_i^{v_i} \quad \textit{for} \quad v_i \in gp\{\theta a_1, \ldots, \theta a_n\} ;$$

in particular, v_i *belongs to* N *, the normal closure of the relators.*

The proof is clear from the definition of Peiffer exchange. Note that each a_i is of the form $u_i^{-1} r_i^{\varepsilon_i} u_i$, $r_i \in R$, $u_i \in F$, and so the subgroup of F generated by the $\theta a_1, \ldots, \theta a_n$ is a subgroup of N .

The Peiffer equivalences turn out to be particularly relevant for a class of presentations called 'aspherical' (§7). The groups of such presentations are torsion-free. A wider class of groups and presentations can be discussed using a larger class of operations than the Peiffer equivalences - for example, in this way one studies the 'combinatorially aspherical' presentations (§8); these determine groups among which are one-relator groups, most Fuchsian groups, and many others. The definition of this wider class of operations is as follows.

Operations on Y-sequences:

(i) *Exchanges* will be the Peiffer exchanges as above.

(ii) A *deletion* is a deletion of an adjacent pair (a, b) in a Y-sequence in case $(\theta a)(\theta b) = 1$ in F . A *collapse* is a sequence of exchanges and deletions in some order.

(iii) An *insertion* is the inverse of a deletion and an *expansion* is the inverse of a collapse. (But note that to insert (a, b) in a Y-sequence we must have not only $(\theta a)(\theta b) = 1$ but also $a , b \in Y \cup Y^{-1}$, so that we still have a Y-sequence.)

(iv) An *equivalence* of Y-sequences is a sequence of collapses and expansions, in some order.

Clearly Peiffer equivalence implies equivalence; it is useful to know when the converse holds.

We say that the presentation $P = (X; R, w)$ is *redundant* if (i) there is a τ in R such that $w\tau = 1$, or (ii) there are ρ, σ in R such that $\rho \neq \sigma$ but $w\rho$ is conjugate to $w\sigma$ or to $w\sigma^{-1}$. (If P is not redundant, it is *irredundant*.) If P is redundant, we can find a, b in $Y \cup Y^{-1}$ such that $(\theta a)(\theta b) = 1$ but $b \neq a^{-1}$; so in this case, an insertion or deletion for a Y-sequence need not be a Peiffer insertion or Peiffer deletion.

We say the presentation P is *primary* if for all $\rho \in R$, $w\rho$ is not a proper power. If this does not hold, then, by Example 2 of §1, we can again find an insertion (or deletion) which is not a Peiffer insertion (or Peiffer deletion).

PROPOSITION 13. *Let* $P = (X; R, w)$ *be a presentation which is irredundant and primary. Then any deletion (insertion) has the same effect as a suitable Peiffer deletion (Peiffer insertion), combined with a sequence of elementary exchanges.*

Proof. Suppose given $a = (\rho^u)^\varepsilon$, $b = (\sigma^v)^\eta$, elements of $Y \cup Y^{-1}$, such that $(\theta a)(\theta b) = 1$. Let $r = w\rho$, $s = w\sigma$. With this notation, we have the following lemma.

LEMMA. *If* P *is irredundant, then* $\rho = \sigma$, $r = s$, $\varepsilon + \eta = 0$ *and for some* $m \in \mathbb{Z}$, $uv^{-1} = z^m$, *where* z *is the root of* r.

The proof of the lemma is easy. We are given $r^\varepsilon = uv^{-1}s^{-\eta}vu^{-1}$. By irredundancy, $\rho = \sigma$, $r = s$ and hence (since $r \neq 1$), $\varepsilon + \eta = 0$. So uv^{-1} centralises r, which implies the lemma.

Since also P is primary, we have further that $z = r$. So

$$a = (\rho^u)^\varepsilon, \; b = (\rho^v)^{-\varepsilon} \text{ with } \varepsilon = \pm 1, \; uv^{-1} = r^m .$$

We now do an elementary exchange of (a, b) to (a_1, b_1) say, where $a_1 = (\rho^{u_1})^{-\varepsilon}$, $b_1 = (\rho^{v_1})^{\varepsilon}$. If $|m + \varepsilon| < |m|$, we use here the elementary exchange $(a, b) \sim (b^{\theta a^{-1}}, a)$ and obtain easily that $u_1v_1^{-1} = r^{-(m+\varepsilon)}$. If $|m - \varepsilon| < |m|$, we use the elementary exchange $(a, b) \sim (b, a^{\theta b})$ and obtain that

$u_1 v_1^{-1} = r^{-(m-\varepsilon)}$. Hence a sequence of $|m|$ elementary exchanges carries (a, b) to an identity sequence (a_m, b_m) with $a_m = b_m^{-1}$ (as elements of $Y \cup Y^{-1}$) . This clearly implies the assertion. □

COROLLARY. *Let* $P = (X; R, w)$ *be a presentation which is irredundant and primary. Then two* Y-*sequences determine the same element of the free crossed* F-*module* $C = H/P$ *if and only if they are equivalent.* □

REMARK. Let $P = (X; R, w)$ be a presentation and let $R = w(R)$, $F = F(X)$ as usual. It is common in the literature to consider not the Y-sequences in the above (where $Y = R \times F$) but what we could call the R^F-sequences $\underset{\sim}{p} = (p_1, \ldots, p_n)$ where each p_i is a conjugate of a relator or its inverse, so that each p_i is an element of $N = N(R)$. If $\underset{\sim}{y} = (a_1, \ldots, a_n)$ is a Y-sequence, then $\theta'\underset{\sim}{y} = (\theta a_1, \ldots, \theta a_n)$ is an R^F-sequence. We say $\underset{\sim}{p}$ is an *identity* R^F-sequence if $p_1 \ldots p_n = 1$ in F . Clearly $\underset{\sim}{y}$ is an identity Y-sequence if and only if $\theta'\underset{\sim}{y}$ is an identity R^F-sequence. It is these identity R^F-sequences which are considered in [L-S] and [C-C-H] . The operations on Y-sequences can be modelled in R^F-sequences. The *elementary exchanges* replace an adjacent (p, q) in $\underset{\sim}{p}$ by (q, p^q) or $(q^{p^{-1}}, p)$; these are called *exchanges* in [C-C-H] and *Peiffer transformations of the first kind* in [L-S] . The *deletions* or *insertions* delete or insert an adjacent (p, p^{-1}) . (The deletions are called *Peiffer transformations of the second kind* in [L-S], and insertions are not considered. Both operations are considered in [C-C-H].) *Equivalences* of R^F-sequences are composites of exchanges, deletions and insertions, in some order. Clearly the map θ' from Y-sequences to R^F-sequences induces a bijection of equivalence classes. However, for R^F-sequences there is no notion corresponding to our Peiffer equivalence, and so in general we do not recover the free crossed F-module (C, ∂) of P from the R^F-sequences. Nonetheless by the last Corollary, we may recover C , and hence the module π of identities, from the R^F-sequences if P is irredundant and primary.

7. Aspherical 2-complexes and aspherical presentations

A topological space X is *aspherical* if it is connected and $\pi_i X = 0$ for $i > 1$. Thus for such X the significant homotopy invariant is the fundamental group $\pi_1 X$, and for aspherical CW-complexes X the fundamental group determines the homotopy type of X. (See for example [W].) If K is a connected 2-dimensional CW-complex, then K is aspherical if and only if $\pi_2 K = 0$.

PROPOSITION 14. *Let* $K = K(P)$ *be the geometric realisation of a presentation* $P = (X; R, w)$. *Then the following are equivalent.*

(i) *The 2-complex* K *is aspherical, i.e.* $\pi_2 K = 0$.

(ii) *The module* π *of identities for* P *is zero.*

(iii) *The relation module* $\bar{N}$ *of* P *is the free module on the induced map* $\bar{w}: R \longrightarrow \bar{N}$.

(iv) *Any identity* Y*-sequence for* P *is Peiffer equivalent to the empty sequence.*

Proof. That (i) and (ii) are equivalent is immediate from Corollary 2 of Proposition 9. That (ii) and (iii) are equivalent follows from the Corollary to Proposition 7. Finally, the equivalence of (ii) and (iv) follows from Proposition 11. □

We now follow [T, S1, C-C-H] in calling a presentation P *aspherical* if $\pi_2 K(P) = 0$, i.e. if any of the equivalent properties of Proposition 14 hold. There is another useful condition for P to be aspherical.

PROPOSITION 15. *A presentation* $P = (X; R, w)$ *is aspherical if and only if* P *is irredundant and primary, and any identity* Y*-sequence for* P *is equivalent to the empty sequence* $\emptyset$.

Proof. Suppose first that P is aspherical. We prove that P is irredundant and primary.

Let $\rho \in R$. Since $\pi = 0$, $d_2 e^2_\rho \neq 0$ and so $w\rho \neq 1$. Let $\sigma \in R$ and suppose $r = w\rho$ is conjugate to $s = w\sigma$, i.e. $r = u^{-1} s u$ for some $u \in F$. Then the elements ρ, σ^u of the free crossed F-module C of P satisfy $\partial\rho = \partial\sigma^u$. Since $\operatorname{Ker}\partial = \pi = 0$, we have $\rho = \sigma^u$ in C and so $e^2_\rho = e^2_\sigma . \phi u$ in $C_2(P)$. By freeness of $C_2(P)$, $\rho = \sigma$. A similar proof, with e^2_σ replaced by $-e^2_\sigma$, shows that r cannot equal $u^{-1} s^{-1} u$.

Suppose now $r = z^q$ where $q \geq 1$. Then $r = z^{-1} r z$. The above proof shows that $e^2_\rho = e^2_\rho . \phi z$ and so $\phi z = 1$. Hence

$z = \partial a$ for some $a \in C$. Then $\partial\rho = \partial a^q$. Since $\operatorname{Ker} \partial = 0$, $\rho = a^q$ and therefore in $C_2(P)$, e^2_ρ is divisible by q. Since e^2_ρ is a basis element for $C_2(P)$, we have $q = 1$.

This completes the proof that if P is aspherical then it is irredundant and primary. The remaining assertions of the Proposition follow from the Corollary to Proposition 13, and Proposition 14. □

8. The identity property

In this section, we present a property of a presentation first described by Lyndon in [L1] and later called the *identity property* by Papakyriakopoulos in [P2]. This property provides a useful characterisation of those presentations which are irredundant and for which any identity sequence is equivalent to the empty sequence. In this section, we abbreviate "Y-sequence" to "sequence".

DEFINITION. Let $P = (X; R, w)$ be a presentation, and let $\underset{\sim}{y} = (a_1, \ldots, a_n)$, where each $a_i = (\rho_i^{u_i})^{\varepsilon_i}$, $\rho_i \in R$, $u_i \in F$, $\varepsilon_i = \pm 1$, be an identity sequence for P. We say $\underset{\sim}{y}$ has the *identity property* if the indices $1, \ldots, n$ can be grouped into pairs (i, j) such that $\rho_i = \rho_j$, $\varepsilon_i + \varepsilon_j = 0$ and, if z_i is the root of $r_i = w\rho_i$, then for some $m_i \in \mathbb{Z}$

$$u_i \equiv z_i^{m_i} u_j \bmod N . \tag{8.1}$$

We say $\underset{\sim}{y}$ has the *primary identity property* if it has the identity property but with (8.1) replaced by

$$u_i \equiv u_j \bmod N . \tag{8.2}$$

We say P has the *(primary) identity property* if every identity sequence for P has this property.

PROPOSITION 16. *Let* P *be a presentation and let* $\underset{\sim}{y}$ *be an identity sequence for* P.

(i) *If* $\underset{\sim}{y}$ *has the identity property then* $\underset{\sim}{y}$ *is equivalent to the empty sequence.*

(ii) *If* $\underset{\sim}{y}$ *has the primary identity property, then* $\underset{\sim}{y}$ *is Peiffer equivalent to the empty sequence.*

Also, the converse to (ii) *holds, and the converse to* (i) *holds if* P *is irredundant.*

Proof. (i) By exchanges we can transform $\underset{\sim}{y}$ to $\underset{\sim}{z}$ which again has the identity property but with adjacent indices paired. Thus we can write $\underset{\sim}{z} = (b_1', b_1, b_2', b_2, \ldots)$ where

$$b_i' = (\rho_i^{u_i'})^{\varepsilon_i}, \; b_i = (\rho_i^{u_i})^{-\varepsilon_i} \text{ and } u_i' = z_i^{m_i} u_i v_i$$

where z_i centralises $r_i = w\rho_i$, and v_i belongs to N. Let $c_i = (\rho_i^{u_i v_i})^{\varepsilon_i}$. Then $\theta c_i = \theta b_i'$, and so by deletions and insertions we can transform $\underset{\sim}{z}$ to $\underset{\sim}{w} = (c_1, b_1, c_2, b_2, \ldots)$. Now $v_i = \partial h_i$ for some h_i in the free crossed module C of P. Hence the product

$$w = c_1 b_1 c_2 b_2 \ldots = h_1^{-1} b_1^{-1} h_1 b_1 h_2^{-1} b_2^{-1} h_2 b_2 \ldots \in [C, C] .$$

But $\underset{\sim}{w}$ is an identity sequence, so $\partial w = 1$. By Proposition 4, $w = 1$ in C. By Proposition 12, $\underset{\sim}{w}$ is Peiffer equivalent to $\emptyset$. Hence $\underset{\sim}{y}$ is equivalent to $\emptyset$.

(ii) This is proved as for (i), but with $c_i = b_i'$, so that $\underset{\sim}{y}$ is Peiffer equivalent to $\underset{\sim}{w}$ and hence to $\emptyset$.

The converse to (ii) holds, since the empty sequence has the primary identity property, and this property is preserved under Peiffer equivalence. A similar reasoning gives the converse to (i), if P is irredundant. □

We now give another characterisation of the identity property for an identity sequence. This will lead to a characterisation of the identity property for a presentation in terms of the structure of the module of identities or, equivalently, the structure of the relation module.

Let $P = (X; R, w)$ be a presentation of a group G. Recall from §§1, 3 and 5 that we have exact sequences

$$1 \longrightarrow E \longrightarrow H \xrightarrow{\theta} F \xrightarrow{\phi} G \longrightarrow 1 ,$$
$$1 \longrightarrow P \longrightarrow H \longrightarrow C \longrightarrow 1 ,$$
$$1 \longrightarrow P \longrightarrow E \longrightarrow \pi \longrightarrow 1 ,$$
$$1 \longrightarrow \pi \longrightarrow C \longrightarrow N \longrightarrow 1 ,$$
$$1 \longrightarrow \pi \longrightarrow \bar{C} \longrightarrow \bar{N} \longrightarrow 1 ,$$

the last of which identifies the module $\pi = E/P$, of identities for P, with a submodule of $\bar{C} = C_2(P)$, the free $\mathbb{Z}G$-module with basis $\{e_\rho^2 \; \rho \in R\}$.

We now construct a module associated with the roots of the relators of the presentation P. Let $\tilde{P}$ be the normal closure in H of the Peiffer group P together with the elements

$$(\rho(\rho^z)^{-1})^u \quad u \in F, \ \rho \in R$$

where z is the root of $w\rho$. Notice that $\tilde{P}$ is an F-subgroup of H and lies in E. Hence $\tilde{P}/P$ is a G-submodule of π and the injection $\pi \to \bar{C}$ identifies $\tilde{P}/P$ with the submodule M of $\bar{C}$ generated by the elements

$$e_\rho^2 \cdot (1 - \phi z) \ , \ \rho \in R \ .$$

We call this submodule the *root module* of P.

PROPOSITION 17. *Let* $\underset{\sim}{y} = (a_1, \ldots, a_n)$ *be an identity sequence for the presentation* P. *Then the following conditions are equivalent.*

(i) $\underset{\sim}{y}$ *has the identity property*

(ii) The element $y = a_1 \ldots a_n$ *belongs to* $\tilde{P}$.

(iii) The image $\bar{y}$ *of* y *in* $\bar{C}$ *is an element of the root module of* P.

Proof. That (ii) $\Longleftrightarrow$ (iii) follows from the identification of $\tilde{P}/P$ with M.

(i) $\Longrightarrow$ (iii) The pairings given by the identity property imply that $\bar{y}$ is a sum of elements of the form

$$\pm e_\rho^2 \cdot (1 - \phi z^{\pm m}) \ \phi u$$

where $u \in F$, z is the root of $w\rho$ and $m > 0$. The rules

$$(1 - \phi z^m) = (1 - \phi z)(1 + \phi z + \ldots + \phi z^{m-1})$$
$$(1 - \phi z^{-m}) = -(1 - \phi z^m)(\phi z^{-m})$$

now imply that $\bar{y}$ belongs to the root module of P.

(ii) $\Longrightarrow$ (i) We are given $y \in \tilde{P}$. Then the image $\bar{y}$ of y in π is also the image of an element $p_1 q_1 \ldots p_\ell q_\ell$ of $\tilde{P}$ where the p_i, q_i are respectively of the form $(\rho^u)^\varepsilon$, $(\rho^{zu})^{-\varepsilon}$ where $u \in F$, $\varepsilon = \pm 1$ and z is the root of $w\rho$. So $\underset{\sim}{y}$ is Peiffer equivalent to the sequence $(p_1, q_1, \ldots, p_\ell, q_\ell)$. This sequence has the identity property, and this property is preserved under Peiffer equivalence. Hence $\underset{\sim}{y}$ has the identity property. □

COROLLARY. *Let* $P = (X; R, w)$ *be a presentation of a group* G. *Then the following are equivalent:*

(i) P *has the identity property.*

(ii) P *is irredundant, and each identity sequence for* P *is equivalent to the empty sequence.*

(iii) P *is irredundant, and the root module of* P *coincides with the module of identities for* P .

(iv) *The relation module* $\bar{N}$ *of* P *decomposes, as a* $\mathbb{Z}G$-*module, into a direct sum of cyclic submodules* $\bar{N}_\rho$, $\rho \in R$, *where each* $\bar{N}_\rho$ *is generated by the image* $\bar{r}$ *in* $\bar{N}$ *of the relator* $r = w\rho$, *subject to the single relation*

$$\bar{r} \cdot (1 - \phi(z)) = 0 ,$$

z *being the root of* r . □

Since (i) and (iv) clearly imply P irredundant, this is immediate from previous results. Notice that condition (iv) says simply that the map $d_2 \colon C_2(P) \longrightarrow C_1(P)$ determines its image $\bar{N}$ as the quotient of $C_2(P)$ by the root module M of P , whence it is clear that (iii) and (iv) are equivalent. Furthermore, it is straightforward to check directly that the identity property implies (iv) .

REMARK 1. Proposition 17 seems to be new. The fact that the identity property for P implies condition (iv) was indicated in [L1] ; that (iv) implies (ii) is due to Huebschmann [Hu3] . In [Hu2] the determination of the module of identities as what we have called the root module is given. A proof that (iv) implies the identity property does not seem to have been given in the literature.

REMARK 2. Various other notions of asphericity for a presentation P are considered in [C-C-H] . These are as follows:

(DA) P is *diagrammatically aspherical* if every identity R^F-sequence over P can be transformed to the empty (identity) sequence by collapses.

(SA) P is *singularly aspherical* if it is diagrammatically aspherical, irredundant and primary.

We note in passing that, in view of Propositions 13, 14 and 15, P is singularly aspherical if and only if every identity Y-sequence over P can be transformed to the empty sequence by Peiffer collapses.

For the next two definitions, note that for any presentation $P = (X; R, w)$, we can find a subpresentation $\hat{P} = (X; \hat{R}, \hat{w})$ of the same group, with $\hat{R}$ contained in R , $\hat{w}$ equal to the restriction of w , and such that $\hat{P}$ is irredundant. We call $\hat{P}$ an *irredundant part* of P .

(CA) P is *combinatorially aspherical* if, for no $\rho \in R$, $w\rho = 1 \in F$ and if P has an irredundant part satisfying one (and hence each) of the four equivalent conditions of the above Corollary.

(CLA) P is *Cohen-Lyndon aspherical* if, for no $\rho \in R$, $w\rho = 1 \in F$ and if P has an irredundant part $\hat{P} = (X; \hat{R}, \hat{w})$ such that the normal closure $N = N(R) = N(\hat{R})$ of $\hat{w}(\hat{R})$ and $w(R)$ in F has a basis

$$B = \bigcup_{\hat{R}} \{uru^{-1} \; ; \; u \in U(r)\}$$

where, for each $r \in \hat{R}$, $U(r)$ is a full left transversal for $NC(r)$, $C(r)$ being the centraliser of r in F.

We note that the definitions given in [C-C-H] differ from the above ones (but are, of course, equivalent).

These notions are linked by the implications

$$\begin{array}{ccccc} & & SA & \Longrightarrow & \text{Aspherical} \\ & & \Downarrow & & \Downarrow \\ CLA & \Longrightarrow & DA & \Longrightarrow & CA \end{array}$$

and are studied extensively in [C-C-H]. The homotopy type of the geometric realisation of a combinatorially aspherical presentation is determined in [Hu2]. In [C-H], diagrammatically aspherical presentations are studied from a geometric point of view, and a consequence of the main result in [C-H] is that small cancellation presentations are diagrammatically aspherical. This was claimed (though in a different terminology) in the proof of Theorem III of [L4], but the proof is not correct.

9. Examples and an unsettled problem of J.H.C. Whitehead.

We now consider examples from §1 in the light of later sections. As explained in §6, for non-primary presentations we must distinguish between Y-sequences and R^F-sequences, and thus the intuitive terminology of §1 is not accurate. In the case of presentations which are irredundant and primary, the distinction between the two kinds of identity sequences is not crucial.

Example 3 of §1 was an identity between six commutators in the generators x, y, z. Suppose for precision that the presentation is irredundant with set R of relators consisting solely of the commutators [x, y], [y, z], [z, x]. Then we have the identity R^F-sequence over $P = (x, y, z; r_1, r_2, r_3)$

$$\underset{\sim}{p} = (r_1, r_2^y, r_3, r_4^z, r_5, r_6^x)$$

where $r_1 = [x, y]$, $r_2 = [x, z]$, $r_3 = [y, z]$, $r_4 = [y, x]$, $r_5 = [z, x]$, $r_6 = [z, y]$. So we have $r_1 = r_4^{-1}$, $r_2 = r_5^{-1}$, $r_3 = r_6^{-1}$. Note that x , y , z do not belong to $N(R)$. So $\underset{\sim}{p}$ does not have the identity property and (since P is irredundant and primary) we may deduce that $\underset{\sim}{p}$ represents a non-trivial element of the module π of identities.

Example 4 of §1 was an identity among relations for the standard presentation of $\mathbb{Z}_2 \times \mathbb{Z}_2$. This presentation is not primary, so we must deal with Y-sequences, and the identity Y-sequence for this example is

$$\underset{\sim}{p} = (\rho, \tau, \sigma^{xy}, (\rho^{-1})^y, \sigma^{-1}, (\tau^{-1})^{\widetilde{x}\widetilde{y}}) ,$$

where ρ, σ, τ are elements of R mapped to r , s , t . Note that $\widetilde{x}\widetilde{y} \notin N(R)$, so that $\underset{\sim}{p}$ does not have the identity property. Hence the corresponding element of $\bar{C}$

$$e_\rho^2(1 - \phi y) + e_\sigma^2(\phi(xy) - 1) + e_\tau^2(1 - \phi(\widetilde{x}\widetilde{y}))$$

does not belong to the root module; this may be verified directly.

In Example 5 of §1 (which illustrates an identity for the standard presentation of the symmetric group S_3) , the relator t occurs three times, and so the corresponding identity sequence will not have the identity property.

An important theorem of Lyndon [L1] is that any one-relator presentation has the identity property. In particular, if $P = (X; r)$ is a one-relator presentation, and P is primary, then P is aspherical. Another proof of this result is given by [D-V], who also construct other examples of aspherical spaces. In particular, they solve a problem of Papakyriakopoulos [P2] in showing that if P is the presentation

$$(a, b, x_1, y_1, \ldots, x_n, y_n; [a, b] \prod_{i=1}^{n} [x_i, y_i], [a, b\tau])$$

where τ belongs to the commutator subgroup $[FX, FX]$, then P is aspherical; the proof is a delicate combination of rewriting arguments and covering space techniques.

Lyndon's theorem is often called the Simple Identity Theorem. A geometric proof of a stronger theorem is given by Huebschmann in [Hu1]; it uses "pictures" (which are described in the next section).

A stronger result again is that a one-relator presentation is

CLA (§8). This result is due to Cohen-Lyndon [C-L] and is reproved in [C-C-H] .

A deep geometric result of Papakyriakopoulos [P1] (the sphere theorem) implies the asphericity of certain presentations of the groups of knots and of links. Here a *link group* is the fundamental group of the complement of a tame link L in S^3 . The link L is called *geometrically unsplittable* if there is no embedded pℓ 2-sphere S^2 in $S^3\backslash L$ such that each component of $S^3\backslash S^2$ contains points of L .

THEOREM [P1] : *If* L *is a link in* S^3*, then* $S^3\backslash L$ *is aspherical if and only if* L *is geometrically unsplittable.*

Now the link group $G = \pi_1(S^3\backslash L)$ has the *Wirtinger presentation* $P = (x_1, \ldots, x_n; r_1, \ldots, r_m)$ coming from an oriented diagram for L , with generators $x_1, \ldots, x_n$, one for each overpass, and for each crossing

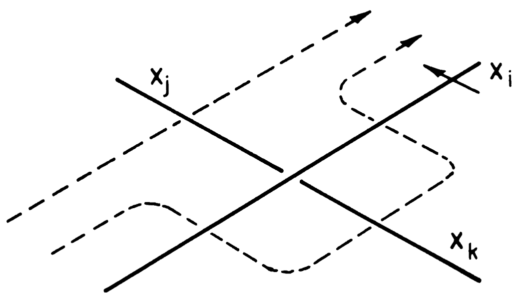


Figure 3

a relation $x_j^{-1}x_i^{-1}x_kx_i = 1$ (corresponding to a deformation between the two dotted paths shown in Fig. 3). There is always an identity among these relations, corresponding to a loop drawn right round the diagram of the link. So we have a presentation $P' = (x_1, \ldots, x_n; r_1, \ldots, r_{m-1})$ of the same group. An implication of Papakyriakopoulos' theorem is that P' is aspherical if and only if L is geometrically unsplittable. In particular if L is a knot, then P' is aspherical. For this *asphericity of knots* no purely algebraic proof is known.

In [C-C-H] it is proved, without using the sphere theorem, that the group of any tame graph has a CLA presentation corresponding to a handle decomposition of the exterior space. An immediate consequence is the asphericity of geometrically unsplittable tame graphs.

The asphericity of knots could be easily proved if one knew that the primary identity property is *hereditary*, i.e. is inherited by subpresentations. Here if $P = (X; R)$ is a presentation, then a *subpresentation* is a group presentation $P' = (X'; R')$ for which $X' \subset X$, $R' \subset R$. Of course the group G' of P' may be quite different from G. However it is not even known if the primary identity property is inherited by P' in the case when $X' = X$ and R has one more element than R'; indeed the general finite case would follow from the special case.

The way the asphericity of knots could be deduced from this result is as follows. Add to the Wirtinger presentation of the knot the extra relation x_1. Geometrically, this corresponds to cutting the knot at the overpass x_1. The knot can then be untied. This corresponds to a combinatorial equivalence:

$$(x_1, .., x_n; r_1, .., r_{n-1}, x_1) \sim (x_1, .., x_n; x_1, .., x_n) .$$

Such equivalences preserve the module of identities, and the last presentation clearly has trivial module of identities.

The question of the hereditability of the primary identity property is equivalent to a famous question of Whitehead, raised in [Wh1]: *is every subcomplex of a 2-dimensional aspherical complex aspherical?* This seems a very difficult question: work has been done by [A, Be, B-D, C2, Co, H, Hul, P2, S2, St].

The problem is equivalent to the following. Let L be a connected 2-dimensional complex with $\pi_2 L \neq 0$. Let K be formed from L by attaching a set of 2-cells. Is it true that $\pi_2 K \neq 0$? (For, it is easily seen that attaching any set of 0-cells or 1-cells leaves π_2 non-zero.)

Adams [A] shows that the condition that L be 2-dimensional is essential here. He sets $L = (S^1 \vee S^2) \cup_f e^3$; here π_1 is infinite cyclic generated by z say, $\pi_2 L^2$ is isomorphic to the group ring of $\mathbb{Z}$ and f represents the element $2 - z$ of this group ring. Let $K = L \cup_g e^2$, where g represents the class z. Then $\pi_2 K = 0$ but $\pi_2 L$ is isomorphic with the additive group of fractions $m/2^n$, and so is non-zero.

Another possible generalisation of the question is: can π_n

of an n-complex be killed by attaching n-cells? This is easily settled if $n > 2$. For example if $K = E^3 \vee E^3$ and L is the 3-subcomplex $L = \dot{E}^3 \vee E^3$, then $\pi_3 L = \mathbb{Z}$ but $\pi_3 K = 0$. Thus the question is very much a two-dimensional one, and its difficulties are connected with our lack of understanding of crossed modules, and in particular, of free crossed modules.

In the next section, and in [St], geometric reasons are given which indicate the complexity of the problem.

The book by Lyndon and Schupp [L-S] uses the term *aspherical* in various senses. In referring to the asphericity of knots (p.162) the term is used in the sense given above. On p.157, the term is used to mean, in our terminology, that any identity R^F-sequence collapses to the empty sequence $\emptyset$, i.e. that the presentation is diagrammatically aspherical. However, the presentation $(x; x^2)$ of the group $\mathbb{Z}_2$ is diagrammatically aspherical but not aspherical. Further, it is not true that for any identity sequence $\underset{\sim}{y}$, "$\underset{\sim}{y}$ is equivalent to $\emptyset$" implies "$\underset{\sim}{y}$ collapses to $\emptyset$", nor is it true that "$\underset{\sim}{y}$ is Peiffer equivalent to $\emptyset$" implies "$\underset{\sim}{y}$ Peiffer collapses to $\emptyset$". Examples of this type of phenomenon were given in May, 1978, by J. Howie and by P. Stefan (private communications); later examples were given by I. Chiswell [C-C-H] and by A. Sieradski [S2] for aspherical presentations. Stefan's example is given elsewhere in this volume [St]; Howie's example is the identity sequence

$$\underset{\sim}{c} = ([x, y], [x, z]^y, [y, z], [y, x]^z, [z, x], [z, y]^x)$$

for the presentation $P = (x, y, z; [x, y], [y, z], [z, x], x, y, z)$ of the trivial group. It is easy to see that there must be a Peiffer equivalence of $\underset{\sim}{c}$ to $\emptyset$ (for example, we can check that $\underset{\sim}{c}$ has the primary identity property); a "diagram" for such an equivalence is given in Section 10. However, there is no Peiffer collapse of $\underset{\sim}{c}$ to $\emptyset$ since, by Proposition 12, exchanges on $\underset{\sim}{c}$ turn $[x, y]$ and $[y, x]^z$ to $[x, y]^u$ and $[y, x]^v$ respectively, where u has even length and v has odd length, so that this pair can never be deleted after other exchanges or collapses. Chiswell's example is the sequence

$$(y^{-2}xyx,\ (yx)^{-1}x^{-1}(yx),\ yxy^{-1},\ y(y^{-2}xyx)^{-1}y^{-1})$$

for the presentation $(x, y; y^{-2}xyx, x)$ of the trivial group.

10. Links and pictures

Let K be a two-dimensional CW-complex. In this section we describe some geometric representatives of elements of $\pi_2(K, K^1, a)$, and of homotopies between these representatives.

Suppose the 2-cells of K are indexed by a set R. For each 2-cell e^2_ρ choose a small disc d_ρ inside it. Let f_ρ be the attaching map of e^2_ρ, and choose a path t_ρ joining a point

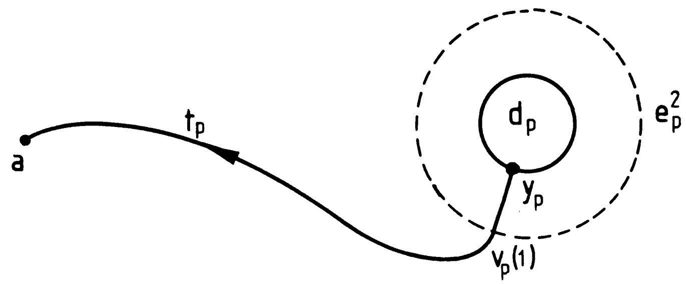


Figure 4

y_ρ of $\dot{d}_\rho$ radially to $f_\rho(1)$, and then joining $f_\rho(1)$ in K^1 to the base point a of K. The characteristic map for e^2_ρ and the path t_ρ together determine an element a_ρ of $\pi_2(K, K^1, a)$, $\rho \in R$.

Let $\alpha \in \pi_2(K, K^1, a)$. Then α is a homotopy class of maps $k: (E^2, S^1, 1) \longrightarrow (K, K^1, a)$. It is a consequence of transversality theory (for more details of which, see [B-R-S], Ch. VII) that α contains a representative k such that for each 2-cell e^2_ρ of K, $k^{-1}(d_\rho)$ (where d_ρ is as above) is a finite disjoint union of discs $\delta_{\rho,1}$, $\delta_{\rho,2}$, ... each of which is mapped by k homeomorphically to d_ρ. (Since E^2 is compact, $k^{-1}(e^2_\rho)$ is non-empty for only finitely many 2-cells e^2_ρ.) For each i, let $x_{\rho,i}$ be the unique point of $\delta_{\rho,i}$ such that $k(x_{\rho,i}) = y_\rho$, and in E^2 join each $x_{\rho,i}$ to 1 by a path $s_{\rho,i}$ so that the various $s_{\rho,i}$ meet each other only at their final point 1, and meet the union of the discs δ only at their initial point. Now relabel the $\delta_{\rho,i}$, $s_{\rho,i}$ as δ_j, s_j, taking the paths in order around 1, and let e^2_ρ be the cell of K containing $k(\delta_j)$. The path $-t_{\rho_j} + k(s_j)$ misses the centres of all e^2_ρ; it therefore can be deformed radially off each e^2_ρ into K^1, so determining a class u_j in $\pi_1(K^1, a)$.

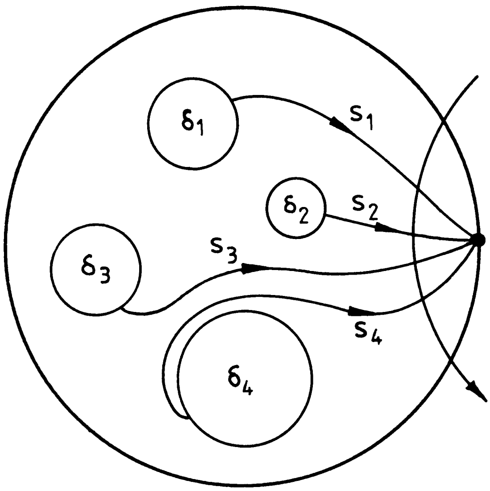


Figure 5

Let ε_j be ± 1 according as k maps δ_j in an orientation preserving or reversing way to $e^2_{\rho_j}$ (the orientations of the 2-cells of K are determined by the characteristic maps and an orientation of the standard 2-cell). Then we have an element

$$c = (\rho_1^{\varepsilon_1})^{u_1} \cdots (\rho_n^{\varepsilon_n})^{u_n}$$

of the free crossed $\pi_1(K^1, a)$-module C.

Let $\phi\colon C \longrightarrow \pi_2(K, K^1, a)$ be the map of crossed $\pi_1(K^1, a)$-modules such that $\phi(\rho) = a_\rho$, $\rho \in R$ (cf. §5). Then if α, k, c are as above, the homotopy addition lemma in dimension 2 implies that $\phi(c) = \alpha$. This explains why ϕ is surjective, a fact used in our proof of the special case of Theorem 10.

Suppose now that $F\colon (E^2, S^1, 1) \times I \longrightarrow (K, K^1, a)$ is a homotopy such that F_0, F_1 satisfy the properties of the map k above. Then F may be deformed rel $E^2 \times \dot{I}$ to a map G so that for all $\rho \in R$, $G^{-1}(d_\rho)$ is a disjoint union of solid tubes $\delta \times I$ and solid tori $\delta \times S^1$ (where δ is a 2-disc), so that G restricted to one of these is projection to δ followed by a homeomorphism to d_ρ. As pointed out in [S2], the union of these $\delta \times I$ and $\delta \times S^1$ for all e^2_ρ is a framed link in $E^2 \times I$. The use of this idea by Whitehead in [Wh1] (cf. [Br1]) suggested to Stefan and to Sieradski [St, S2] a geometric interpretation of Peiffer moves. We illustrate the method using slightly different conventions to those of [S2, St] . (R. Peiffer has informed us that Reidemeister was aware of this interpretation.)

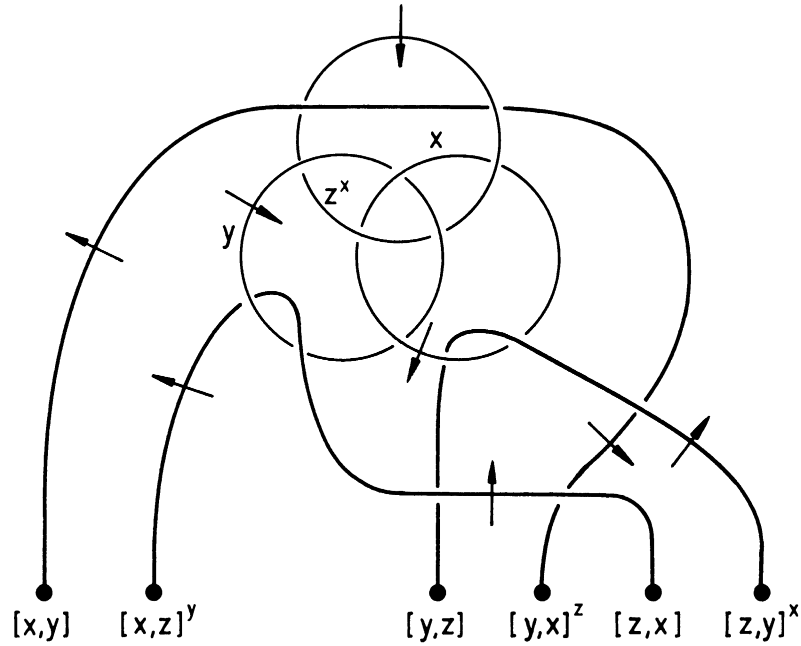
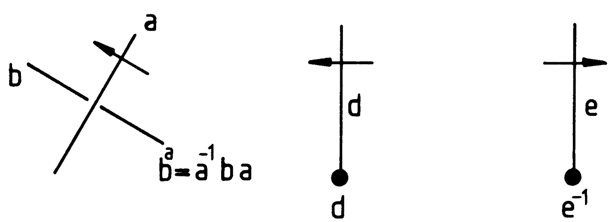


Figure 6

EXAMPLE: The above is a labelled, oriented link diagram with six "feet". Not all the labels have been inserted, because the remaining ones can all be deduced from the rules :



(i) (ii) (iii)

The labels are to be conjugates of relators, considered as elements of the free group FX on the generators. With this convention, the above diagram is consistent, in that no overpass has distinct labels in FX ; the checking of this is left as an exercise to the reader.

The diagram determines an equivalence of $\underset{\approx}{c}$ to $\emptyset$ for the presentation $P = (x, y, z; [x, y], [y, z], [z, x], x, y, z)$ of the trivial group, where $\underset{\approx}{c}$ is the identity sequence given at the base of the diagram. Successive identity sequences in this equivalence are found by horizontal cuts of the diagram, at different heights and in general positions; the identity sequence corresponding to a cut is read off by rules (ii) and (iii) . (The diagram is a simplification by R. Brown of a diagram of an equivalence of $\underset{\approx}{c}$ to $\emptyset$ with about 100 crossings, drawn by J. Howie.)

The reader will note the appearance of the Borromean rings in Fig. 6. They appear for a similar reason in [F-T]. The reader is invited to try and find an equivalence of $\underset{\approx}{c}$ to $\emptyset$ for which the corresponding diagram has the inserted x, y, z less subtly linked.

The above diagram is a partial representation of a null-homotopy $G: (E^2, S^1, 1) \times I \longrightarrow (K, K^1, a)$. We have drawn only the centre lines $0 \times I$, $0 \times S^1$ (where 0 is the centre of δ) of the components of the framed link, and the link itself takes account only of the 2-cells of K and not the 1-cells. Rourke in [Rou] has developed the above use of what is essentially transversality to give a more detailed description of maps $k: (E^2, S^1, 1) \longrightarrow (K, K^1, a)$ in terms of "pictures". We explain the idea for those 2-complexes that are geometric realisations of presentations.

Consider the presentation $(X; R) = (x, y; r, s, t)$ of the trivial group, where $r = x^2y$, $s = y^{-1}x$, $t = x$. Let $K = K(X; R)$ be its geometric realisation, with vertex a . Here is an example of a "picture" of a particular map $k: (E^2, S^1, 1) \longrightarrow (K, K^1, a)$ with $k(S^1) = \{a\}$:

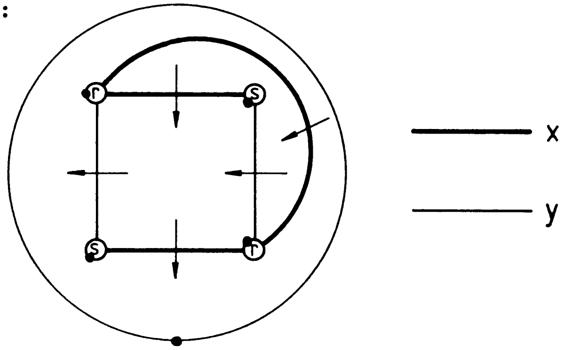


Figure 7

The constituents of such a picture over a presentation $(X; R, w)$ are:

(i) a disc D with boundary ∂D ;

(ii) disjoint discs $\Delta, \Delta_1, \Delta_2, \ldots, \Delta_n$ inside D , each labelled by an element of R or, in the case of an irredundant presentation, by a relator;

(iii) disjoint edges $e, e_1, \ldots, e_m$ inside D and outside the Δ's ; each edge is either a circle, or joins the boundaries of two of the discs in (i) or (ii) ; each edge has a normal orientation, indicated by a short arrow meeting the edge transversally, and is also labelled by a 1-cell of K^1 (identified with a generator of $\pi_1(K^1)$) ;

(iv) base points, indicated by a dot, on the boundaries of D and of each of the Δ's , but not lying on any edge.

The one further condition imposed is that starting from a base point of some Δ and reading the oriented edges round Δ in an anticlockwise direction should give the relator, or its inverse, labelling that disc.

A picture is called *spherical* if it has no edges meeting ∂D ; so the picture of Figure 7 is spherical.

Any picture over a presentation $(X; R, w)$ determines some Y-sequences over $(X; R, w)$ and if the picture is spherical these Y-sequences are identity sequences. We illustrate this process first for the picture of Figure 7.

For each Δ draw a line from the base point of Δ to that of D so that these lines cross the edges transversally and meet only at the base point of D . This gives for example, the next figure:

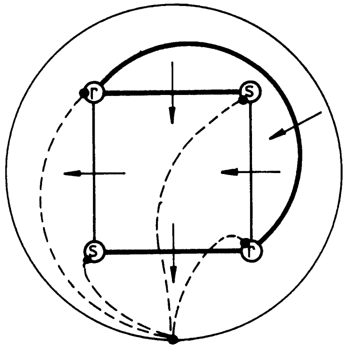


Figure 8

To each dotted line α we can associate a symbol $(\rho^u)^\varepsilon$. Here ρ is the label of the disc from which α starts. This disc also has a sign ε which is $+1$ or -1 according as reading anticlockwise round the disc, starting at the base point, gives $w\rho$ or $(w\rho)^{-1}$. The element u of F is the product of the labels of the edges that α crosses, taken in order from the initial disc of α , and with a sign $+1$ or -1 according as α crosses the edge in a positive or negative normal direction.

The dotted edges have an order, obtained by reading them anticlockwise round the base point of D . From this order, and the associated symbols, we obtain a Y-sequence. This gives for Fig. 8 the identity sequence:

$$\underset{\sim}{y} = (r^x, s^x, s^{-1}, r^{-1}) .$$

The Peiffer transformations now have the following interpretations. A Peiffer insertion, e.g.

$$\underset{\sim}{y} = (r^x, s^x, x^{-1}, r^{-1}) \longmapsto (t^{-1}, t, r^x, s^x, s^{-1}, r^{-1}) = \underset{\sim}{z}$$

corresponds to introducing in Fig. 8 a new component with only two discs as indicated in Fig. 9 (where we now omit to draw ∂D) :

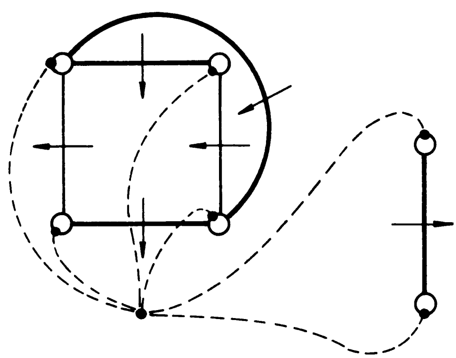


Figure 9

Conversely, we can carry out a deletion on Fig. 9 . (A general description of insertions and deletions is given later.)

An elementary Peiffer exchange, e.g.

$$(t^{-1}, t, r^x, s^x, s^{-1}, r^{-1}) \longmapsto (t^{-1}, r, t, s^x, s^{-1}, r^{-1})$$

corresponds to rechoosing two successive lines from base points of discs Δ to the base point of D . In our example, the result is indicated in Fig. 10.

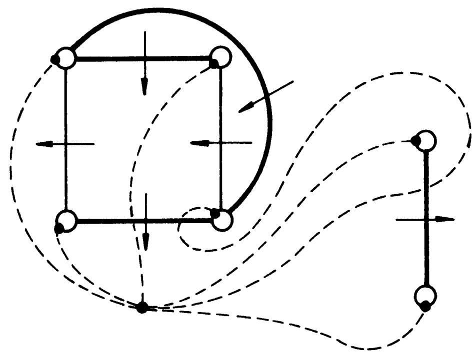


Figure 10

Hence a Peiffer exchange corresponds to rechoosing the paths from the base points of the Δ's to that of D .

The identity sequence $y = (r^x, s^x, s^{-1}, r^{-1})$ given above is P. Stefan's example [St] of an identity sequence that is Peiffer equivalent to $\emptyset$, but does not collapse to $\emptyset$. However, by an insertion y is transformed as above to $\underset{\approx}{z}$, and $\underset{\approx}{z}$ does collapse to $\emptyset$ [St]$^{\sim}$.

We now return to Examples 4, 5 from §1, and show further how to obtain pictures from which we can "read off" the corresponding identity sequence.

EXAMPLE 4. Let K be the realisation of the presentation $(x, y; r, s, t)$ of $\mathbb{Z}_2 \times \mathbb{Z}_2$ given in §1, and let $\tilde{K}$ be the universal cover of K . The 1-dimensional Cayley complex given in Fig. 1 is the 1-skeleton $\tilde{K}^1$ of $\tilde{K}$, and the labels of the edges determine the covering map $\tilde{K}^1 \longrightarrow K^1$. Using Fig. 1 we can regard $\tilde{K}^1$ as contained in S^2 (taken as a disc with boundary identified to a point), and the map $\tilde{K}^1 \longrightarrow K$ extends to a map $f\colon S^2 \longrightarrow K^2$ in which the regions in which $\tilde{K}^1$ divides S^2 are mapped to the labelled cells, the outside of $\tilde{K}^1$ being mapped to t^{-1} . This map corresponds to a spherical picture which arises in essence as dual to the Cayley complex. This picture is given by the thick lines in Fig. 11; the thin lines give the Cayley diagram, the dots denote base points, and the dotted lines are used to determine a Y-sequence from the picture, as described earlier. The resulting Y-sequence is precisely the identity sequence given in §1 .

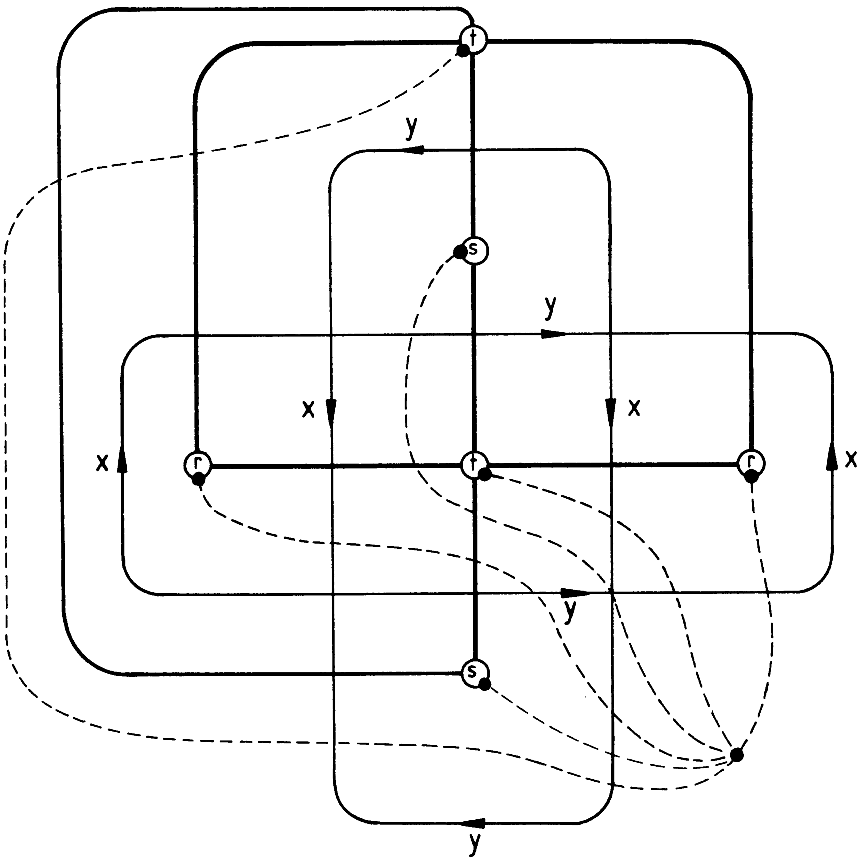


Figure 11

EXAMPLE 5. In this example we give only the picture corresponding to the Cayley diagram, together with the dotted curves which determine the corresponding identity sequence.

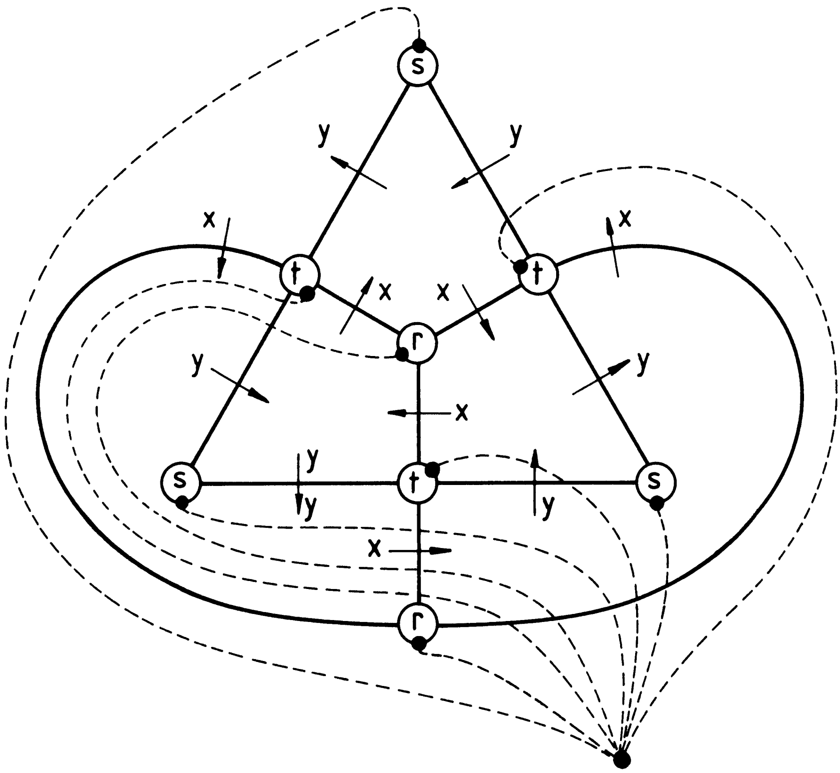
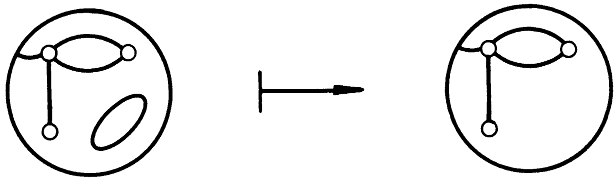


Figure 12

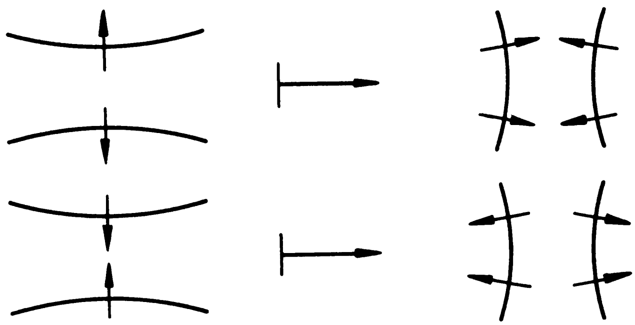
We now sketch the ideas of "deformations" of pictures, which correspond to homotopies. These are:

(D1) Removal of edges which are loops enclosing no discs or edges e.g.

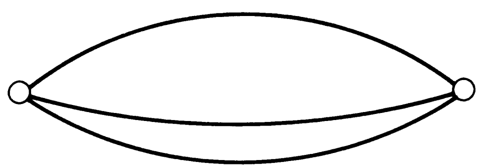


(D1') Insertion of such a 'floating circle'.

(D2) Bridge moves:



(D3) Removal of a 'floating component' enclosing no other discs or edges, such as



(D3') Reverse of (D3) .

The operation (D3) corresponds to a Peiffer deletion, but may be applicable only after a sequence of bridge moves.

These pictures and deformations are exploited by Rourke [Rou] and Huebschmann [Hu1] .

In particular, [Hu1] uses them to give examples of aspherical 2-complexes for which every subcomplex is aspherical. No such families were known before.

The hereditability of the primary identity property, which was discussed in the last section, can be expressed in terms of pictures and deformations as follows. Let $(X'; R')$ be a sub-presentation of the aspherical presentation $(X; R)$. Let P be a picture for an identity sequence $\underset{\sim}{y}$ for $(X'; R')$. Since $(X; R)$ is aspherical, there is a sequence of deformations $P \longmapsto P_1 \longmapsto \dots \longmapsto P_n = \emptyset$, which may involve moves (D1') or (D3') using edges labelled by elements of $X \backslash X'$, or discs labelled by elements of $R \backslash R'$. The hereditability of the primary identity property would imply that there is also a sequence of deformations $P \longmapsto P'_1 \longmapsto \ldots \longmapsto P'_m = \emptyset$ involving labels only from X' and R'.

As one final indication of the difficulty of the area of the homotopy theory of 2-complexes, we mention that Reidemeister's paper [Re1], giving an algebraic description of $\pi_2(K)$, was published in 1934. Fifty-five years later, there still does not seem to be available a general way of calculating $\pi_2(K)$ as a $\pi_1(K)$ module even if $\pi_1(K)$ is some reasonably small finite group G. A simpler question might be to ask: which complex representations of G arise as $\pi_2(K) \otimes \mathbb{C}$ for some geometric realisation K of a finite presentation of G?

The history of the methods described in this article is complex, and is to some extent shown by the references given throughout. In effect, the use of the chains of the universal cover is due to Reidemeister [Re1]. His work and that of his students developed in Eilenberg-MacLane's work on complexes with operators, and, in the hands of J.H.C. Whitehead, into simple homotopy theory and what is now termed algebraic K-theory. Coming to the present field, we should mention the paper [Sch], which contains the result of our §5 that $\operatorname{Im} d_2 \cong \underline{N}$. Note that the description of d_2 in terms of the free differential calculus is given in [Fo]. An equivalent, and earlier, formulation is due to Whitehead, in that Theorem 8 of [Wh3] gives a clear and complete description of the relationship between a free crossed module over a free group and the associated chain complex.

There are two useful generalisations of the embedding of the relation module $\underline{N}$ into the free module $\bigoplus_X \mathbb{Z}G$. One of these, described lucidly in [Cr], assigns to any exact sequence of groups $1 \longrightarrow N \longrightarrow \Gamma \xrightarrow{\phi} G \longrightarrow 1$ an exact sequence of G-modules $0 \longrightarrow \underline{N} \longrightarrow D_\phi \longrightarrow IG \longrightarrow 0$. (This is in fact Satz 15 of [Sch].) Another is the Magnus embedding of a free group into a matrix group [Ma]; this is applied in [B1, D2] and an account of the generalisation to the case of a homomorphism $\phi: \Gamma \longrightarrow G$, is given in [Bi] §3.2, where further references may be found.

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SUMMARY OF NOTATIONS.

$P = (X;\ R,\ w)$ is a presentation of a group G. The sequence

$$1 \longrightarrow E \longrightarrow H \xrightarrow{\theta} F \xrightarrow{\phi} G \longrightarrow 1$$

is exact where $H = F(X \times R)$, $F = F(X)$.
The Peiffer group P is normal in H; $N = \operatorname{Im} \theta$. There is a diagram of short exact sequences

$$\begin{array}{ccccccccc}
 & & 1 & & 1 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & P & = & P & & & & \\
 & & \downarrow & & \downarrow & & & & \\
1 & \longrightarrow & E & \longrightarrow & H & \longrightarrow & N & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & & & \\
1 & \longrightarrow & \pi & \longrightarrow & C & \longrightarrow & N & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & & & \\
 & & 1 & & 1 & & & &
\end{array}$$

The free crossed F-module $C \xrightarrow{\partial} F$ is isomorphic to

$$\pi_2(K,\ K^1) \longrightarrow \pi_1(K^1) \quad \text{where} \quad K = K(P)\ .$$

There is a diagram of G-modules, with exact rows:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \pi & \longrightarrow & \bar{C} & \longrightarrow & \bar{N} & \longrightarrow & 0 \\
 & & & & \downarrow \cong & & \downarrow i & & \\
 & & & & \oplus\ \mathbb{Z} G & \xrightarrow{d_2} & \oplus\ \mathbb{Z} G & \xrightarrow{d_1} & \mathbb{Z} G \\
 & & & & \downarrow \cong & & \downarrow \cong & & \downarrow \cong \\
 & & & & C_2(K) & \longrightarrow & C_1(K) & \longrightarrow & C_0(K)
\end{array}$$

i injective

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Note added in proof: K. Igusa in a preprint on 'The generalised Grassman invariant' also has described pictures like those of §10 and has used these for giving an explicit description of the exotic element in $K_3(\mathbb{Z}) = H_3(St(\mathbb{Z})) = \mathbb{Z}_{43}$.

On Peiffer transformations, link diagrams and a question of J. H. C. Whitehead

P. STEFAN

Introduction by Ronald Brown.

Whitehead's famous question: is every subcomplex of a 2-dimensional aspherical complex aspherical? was discussed at Bangor in March, 1978, during a visit of Johannes Huebschmann. He suggested a possible approach to this question, but doubts were raised about this in May, 1978, in a letter to me from Eldon Dyer. Peter Stefan then pinpointed precisely the failure of the proposed method, by finding an example of an identity sequence which was equivalent to Peiffer transformations to the empty sequence $\emptyset$ but which did not collapse to $\emptyset$. He sent the example to several people, and Roger Fenn in replying explained the method of pictures. Peter wrote to Roger on 17 May, 1978, and circulated this letter. Peter died in a mountaineering accident on June 18.

This note contains Peter's example, and the major part of his letter, omitting some irrelevant matters or outdated points. The article "Identities among relations" by R. Brown and J. Huebschmann, in this volume, and referred to here as [Br-Hu], is intended to give the background required for understanding this note, and so some of Peter's notations, conventions and diagrams have been changed to make the two articles consistent. Other changes are few and minor. References here other than to [Br-Hu] are to the bibliography of that article.

1. A Peiffer trivial identity sequence which does not collapse.

Consider the presentation $(x, y; r, s, t)$ of the trivial group in which $r = x^2y$, $s = y^{-1}x$, $t = x$. Consider the identity sequence

$$\underset{\sim}{u} = (r^x, s^x, s^{-1}, r^{-1}) .$$

CLAIM. There is an equivalence of $\underset{\sim}{u}$ to $\emptyset$, but $\underset{\sim}{u}$ does not collapse to $\emptyset$.

Proof. Note first that $rs = x^3$ which commutes with x . So we have Peiffer transformations (where $\tilde{a}$ denotes a^{-1})

$$(r^x, s^x, s^{-1}, r^{-1}) \mapsto (t^{-1}, t, r^x, s^x, s^{-1}, r^{-1}) \mapsto (t^{-1}, r, t, s^x, s^{-1}, r^{-1})$$

$$\mapsto (t^{-1}, r, s, t, s^{-1}, r^{-1}) \mapsto (t^{-1}, r, s, s^{-1}, t^{\tilde{s}}, r^{-1})$$

$$\mapsto (t^{-1}, r, s, s^{-1}, r^{-1}, t^{\tilde{s}\tilde{r}}) \mapsto (t^{-1}, r, r^{-1}, t^{\tilde{s}\tilde{r}}) \mapsto (t^{-1}, t^{\tilde{s}\tilde{r}}) .$$

However $\tilde{s}\tilde{r} = x^{-3}$. So this last identity sequence is equivalent to $\emptyset$, and we have an equivalence $\underset{\sim}{u} \curlywedge \emptyset$.

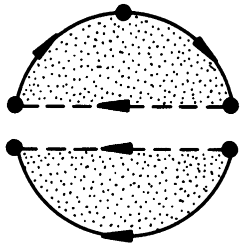
To prove that $\underset{\sim}{u}$ does not collapse to $\emptyset$, note that the presentation is irredundant and primary. In order to obtain a collapse, we must perform Peiffer exchanges on $\underset{\sim}{u}$ to obtain $\underset{\sim}{v}$ of the form $(r^{xa}, r^{-b}, s^{xc}, s^{-d})$ where a, b, c, d are elements of the group $Q = gp\{r^x, s^x, r, s\} = gp\{xyx, s^{-1}y^{-1}x^2, x^2y, y^{-1}x\}$ $= gp\{x^3, x^{-1}y^{-1}x^2, x^2y\}$. Now the centraliser $C(r) = gp\{r\} \subseteq Q$ and Q does not contain x . Hence $xQ \cap C(r) = \emptyset$. But r^{xa} cancels with r^{-b} if and only if $xab^{-1} \in C(r)$, which we have just shown impossible.

2. On Peiffer transformations, and link diagrams.

(Extract from a letter to Roger Fenn.)

The example given before, of a trivial identity sequence which does not collapse, arose in the following way.

The first example to try is $(x; x^2)$, but this does not work on the algebra level. The point is you are working in $F = F(x, y, \ldots)$ and so r^x and r cannot be different if they are given by the same word. In fact any identity sequence (r^u, s^v) $(r, s \in R, u, v \in F)$ collapses by definition of deletion: you are allowed to delete an adjacent pair which cancels in F . So you need at least (r^a, s^b, r^c, s^d) and you also need r not conjugate to $s^{\pm 1}$ because Lyndon proved that a presentation with a *single* relator is necessarily aspherical in the strong sense that every identity among the conjugates of one relator actually collapses to $\emptyset$. (This is a difficult theorem - see Proposition 10.6 and its proof on p. 160 and also the remarks preceding Proposition 11.1 on p. 161 in [L-S] [see also [C-C-H]].) Trying $(x; x^2) \cong (x, y; xy^{-1}, yx)$ fails on this conjugacy condition, so the next simplest thing is to try $(x; x^3) \cong$ $(x, y; x^2y, y^{-1}x)$ and that works.



If I understand your letter correctly, r^x and r *can* represent a different element of $\pi_2 K$ even when equal in F , which shows that Lyndon's 'Peiffer machine' is a bad model for $\pi_2 K$. [This point is about the distinction between Y-sequences and R^F-sequences: see §6 of [Br-Hu]].

A bad thing about our examples is that they seem to depend heavily on the presence of torsion in

$$\pi_1 K = G \cong (X; R) .$$

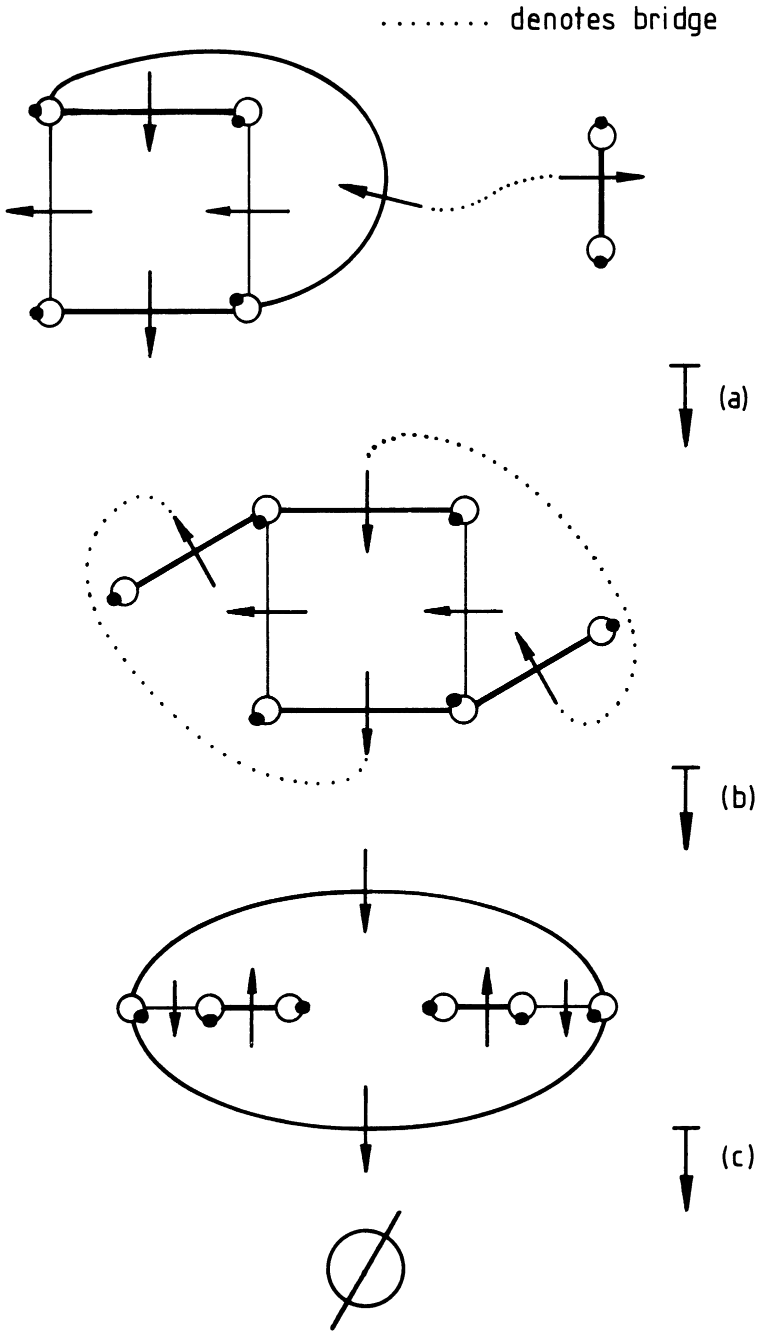
On the other hand $\pi_2 K = 0 \Longrightarrow K = K(G, 1) \Longrightarrow$ cohomological dimension of a subgroup is less than or equal to that of the group, and $\mathbb{Z}_n$ has cohomological dimension ∞ .

I like very much your description of homotopies in terms of bridge moves, but your strip in the first letter can be shortened a bit - only one insertion is needed:

These bridge moves occur in my Peiffer moves as follows:

$$(t^{-1}, r, s, t, s^{-1}, r^{-1}) \xmapsto{\text{(a and b)}} (t^{-1}, r, s, s^{-1}, r^{-1}, t^{\widetilde{\widetilde{sr}}}) \xmapsto{\text{collapse}} \emptyset .$$

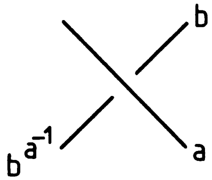
I have a different way of drawing pictures of such homotopies which is simpler but much less useful as it does not carry as much information as your method. Essentially, one ignores the edges of your graphs and worries only about the vertices [i.e. the discs]. This is inspired by Whitehead's approach to $\pi_2(X, X_0)$ where $X = X_0 \cup$ (2-cells) . A neat exposition of Whitehead's proof that $\pi_2(X, X_0)$ is a free crossed $\pi_1 X_0$-module has been written out by Ronald Brown [Brl] . Essentially, the construction is exactly the one explained to me by you and Colin Rourke, except that one does not worry about K^1 and looks only at the pre-image of the mid-points of the 2-cells. So the homotopy is represented by a kind



.... denotes bridge
(a)
(b)
(c)

of "linkage"; the relationship to π_1 is not visible, but somehow it did not matter to Whitehead.

There is a completely *formal* way of getting these linkages out of the Peiffer moves: the two exchanges are represented by



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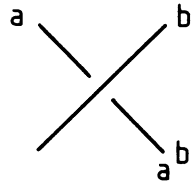
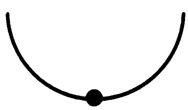


(the line in front represents the element which does not change). The insertions and deletions are represented by



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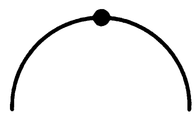
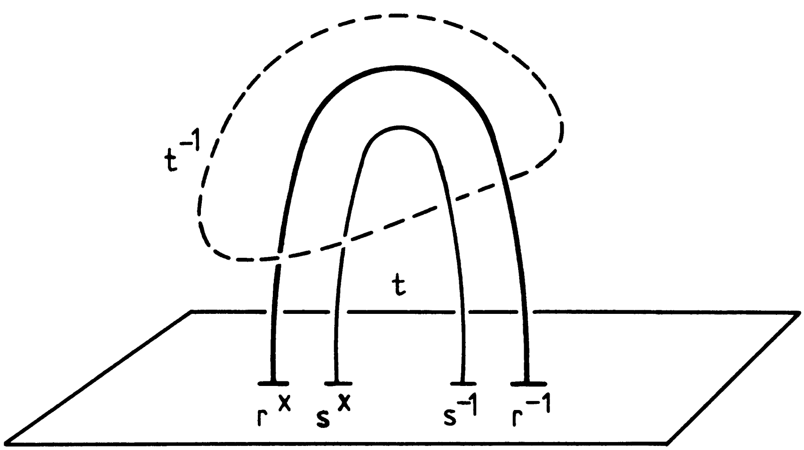
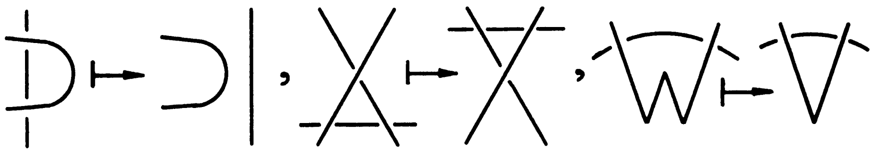


and one reads the diagram upwards. For example, the sequence $(r^x, s^x, r^{-1}) \longmapsto (t^{-1}, t, r^x, s^x, s^{-1}, r^{-1}) \longmapsto \dots \longmapsto \emptyset$ of §1 is given by



The obvious elementary moves are clearly allowed such as



(This is obvious for geometric reasons, but it can be checked directly from the Peiffer transformations.) In contrast to your diagrams, the linkages capture only the formal side of things and are independent of the details of the presentation. In fact, some linkages are impossible - see below. If R is *irredundant* (no $r \in R$ is conjugate in F to $s^{\pm 1}$ for $s \in F$, $s \neq r$) then we may *colour* each strand in the linkage according to the relator r whose conjugates $r^a = a^{-1}ra$, $a \in F$, label it. Furthermore, Peiffer exchanges clearly leave the class mod N of the 'exponent' $a \in F$ invariant, so the colouring may be further refined: each basic colour (one for each $r \in R$) is subdivided into *shades* (one for each class mod N , i.e. one for each element of $G = F/N$) . If, further, F is *primary* (no $r \in R$ is a proper power of some element of F) then the centralizer of each $r \in R$ is simply the cyclic group generated by r and so $C(r) \subset N$. Hence, *if* R *is irredundant and primary, then only strands of the same shade can cancel.*

If now $K = |(X; R)|$ is the 2-dimensional complex corresponding to the presentation $(X; R)$ of G , then it is known that $\pi_2 K = 0$ if and only if the following three conditions hold:

1. R is irredundant,
2. R is primary,
3. The presentation is aspherical in the sense that every n-tuple $\underset{\sim}{p} = (p_1, p_2, \ldots, p_n)$ such that $p_i = (r_i^{\varepsilon_i})^{u_i}$, $r_i \in R$, $u_i \in F$, $\varepsilon_i = \pm 1$ and $p_1 p_2 \ldots p_n = 1$ in F , satisfies $\underset{\sim}{p}$ is equivalent to $\emptyset$ by Peiffer transformations.

Now conjugation in F induces an action of $G = F/N$ on the abelianised group $\overline{N}$ making $\overline{N}$ into a $\mathbb{Z}G$-module. Let $\Delta_r : \mathbb{Z}G \longrightarrow \overline{N}$ be given by $x \longmapsto \overline{r}.x$ where $\overline{r}$ is the class of $r \in R$ in $\overline{N}$. The image of Δ_r is the cyclic submodule $\overline{r}.\mathbb{Z}G$

of $\bar{N}$. Consider the homomorphism of $\mathbb{Z}G$-modules

$$\Delta = \bigoplus_{r\in R} \Delta_r : \bigoplus_{r\in R} \mathbb{Z}G \longrightarrow \bar{N} .$$

Then conditions 1, 2, and 3 above (and so the condition $\pi_2 K = 0$) are equivalent to the requirement that Δ is an isomorphism of $\mathbb{Z}G$-modules. It follows that if $\pi_2 K = 0$ then we have for every $r \in R$ a well defined homomorphism Deg_r of N onto the free abelian group $\mathbb{Z}G$ given by

$$Deg_r : N \xrightarrow{\lambda} \bar{N} \xrightarrow{\Delta^{-1}} \bigoplus_{r\in R} \mathbb{Z}G \xrightarrow{p_r} \mathbb{Z}G$$

where λ is abelianisation and p_r is the r'th projection. Note that Deg_r is defined by the conditions

$$Deg_r((s^k)^a) = k\,[a]\,\delta(r, s)$$

for every $s \in R$, $a \in F$ and integer k ; $[a]$ denotes the image of $a \in F$ in $G = F/N$ and $\delta(r, s)$ is the Kronecker δ . Following Deg_r by the evaluation homomorphism $\mathbb{Z}G \longrightarrow \mathbb{Z}$ at various points of G or by the augmentation homomorphism $\mathbb{Z}G \longrightarrow \mathbb{Z}$ we get various integer valued 'degrees'.

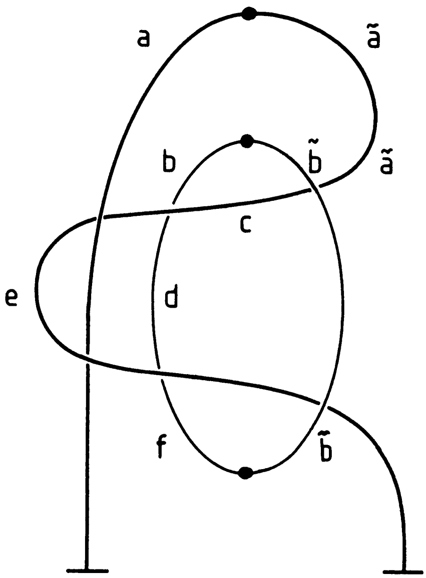
Using these degrees, together with the fact that a centralizer of an element of a free group is necessarily a cyclic subgroup of the free group, we can show that certain linkages are forbidden in an aspherical $K (\pi_2 K = 0)$.

EXAMPLE 1. *(Whitehead linkage)*. Suppose $K = |(X; R)|$ where $\pi_2 K = 0$ and $r, s \in R \subset F$ with $r \neq s$. Let $a = (r^\varepsilon)^u$, $b = (s^\eta)^v$ where $u , v \in F$ and $\varepsilon , \eta = \pm 1$. Write Deg_a for $\varepsilon.Deg_r : N \longrightarrow \mathbb{Z}G$ followed by the evaluation map $\mathbb{Z}G \longrightarrow \mathbb{Z}$ at $[u] \in G$. Let Deg_b be defined similarly in terms of $\varepsilon.Deg_s$ and $[v] \in G$. Let $\tilde{c}$ denote c^{-1} .

Suppose given a 'Whitehead linkage' corresponding to a sequence of Peiffer transformations. Starting at the top we find

$$c = a^{\tilde{b}} \;, \; d = b^c \;, \; e = c^{\tilde{a}} \;, \; f = d^{\tilde{e}} = (b^c)^{\tilde{c}^{\tilde{a}}} .$$

Now $f = b$ implies $c(\tilde{c}^{\tilde{a}})$ commutes with b . As b is a conjugate of $s \in R$, and R is primary, the centralizer of b in F is the cyclic group $< b >$, and so



$$c(c^{a}) = b^{k} \qquad \text{for some } k \in \mathbb{Z},$$

$$\text{or} \quad (\tilde{a}^{\tilde{b}})(a^{\tilde{b}\tilde{a}}) = b^{k} .$$

Applying Deg_b to both sides of this equation, we obtain $k = 0$. Hence

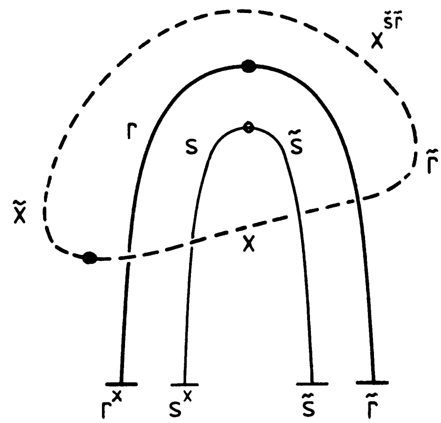
$$(\tilde{a}^{\tilde{b}})(a^{\tilde{b}\tilde{a}}) = 1$$

$$\text{i.e.} \quad a^{\tilde{b}\tilde{a}} = a^{\tilde{b}}$$

$$\text{i.e.} \quad a^{\tilde{b}\tilde{a}b} = a .$$

So $\tilde{b}\tilde{a}b = \tilde{a}^{b}$ commutes with a, and so we obtain $\tilde{a}^{b} = a^{\ell}$ for some $\ell \in \mathbb{Z}$. Taking Deg_a of this equation, we obtain $\ell = -1$ and so b commutes with a. Hence $b = a^{m}$ for some $m \in \mathbb{Z}$. Taking Deg_b we obtain $m = 0$, i.e. $b = 1$, which is a contradiction. Hence the *Whitehead linkage (with different coloured strands) never occurs if* $\pi_2 K = 0$.

EXAMPLE 2. Again assume that $\pi_2 K = 0$. Assume also that $x = (t^{\pm})^{u}$ where $t \in R$ and r, s are conjugates of some relators *other than* t (or conjugates of inverses of such relators). Again rs must commute with x and so $rs = x^{k}$.



Taking Deg_x , we get $k = 0$ or $rs = 1$. Hence the above diagram can be replaced by a simpler one

To prove (or to *disprove*) the Whitehead conjecture it is sufficient to consider the case when the subcomplex L of K differs from K by a single 2-cell. Paint this extra 2-cell red and all the remaining 2-cells blue. Assume that $\pi_2 K = 0$ and let $p = (p_1, p_2, \ldots, p_n)$ be an identity amongst the blue relators (each p_i is a conjugate of a relator in L , or its inverse, and $p_1 p_2 \cdots p_n = 1$ in F) . Then p is equivalent to $\emptyset$ by Peiffer moves in K . This gives a linkage consisting of a blue part X anchored to the 'floor' and a red part, a link Y , floating above. To prove that $\pi_2 L = 0$ we must show that p is equivalent to $\emptyset$ in L - that is that we can get rid of the red stuff. (There is no need to show p collapses to $\emptyset$ - we are not worried about the blue insertions.)

If X and Y are geometrically unlinked, we are done. Otherwise, there seem to be two possible cases:

1. X and Y are algebraically linked as in Example 2. *Part of the problem is to make this idea precise* - perhaps in terms of the various degrees (shades of blue, shades of red).

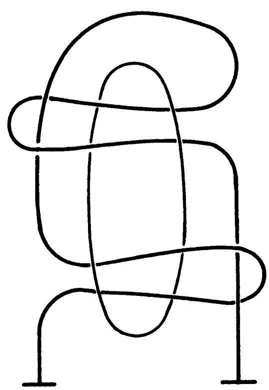
2. X and Y are algebraically unlinked, as in Example 1, but are still geometrically linked.

Now a possible proof (or a search for a counter example) could go as follows (assuming $\pi_2 K = 0$):

(a) In case 1, one would like to show that X and Y can be replaced by X_1 and Y_1 which are 'less linked' algebraically, and so on, ending with X_n and Y_n which are algebraically unlinked.

(b) In case 2, one would like to show that this never happens (for $\pi_2 K = 0$): in other words, the only way to be algebraically unlinked in *aspherical* K is to be also geometrically unlinked.

Unfortunately, the 'degrees' described above do not seem a sufficient tool for all this - they work satisfactorily if there are only two underpasses, but after that they seem to fail. For example, I am unable to show that a 'double Whitehead link' cannot occur if $\pi_2 K = 0$:



It would help a lot if one had some 'higher order degrees' to decide whether a given identity sequence $p = (p_1, \ldots, p_n)$ is equivalent to $\emptyset$. Assume now that R is irredundant and primary, but do *not* assume that (X; R) is aspherical. The first (or zero-order) obstruction is of course $p_1 p_2 \ldots p_n = 1$ in F . Next, the first order obstruction is $\text{Deg}_r p = 0$, for all $r \in R$. (Deg_r is not well-defined on N if (X; R) is not aspherical, but it is still well-defined on n-tuples and also on H (the free group on $F \times R$, see [Br-Hu]).) If all $\mathrm{Deg}_r p = 0$, it is still possible that p is not equivalent to $\emptyset$, and it would be nice to have a second-order obstruction, defined when the first-order one is zero, and so on. It is all a bit reminiscent of higher order linking numbers, and Massey products have been suggested as being relevant.

Here are a few final points;

1. If $\pi_2 K = 0$ and if N is freely generated by a set of conjugates of elements of R , then every identity among relators actually collapses [L-S, p.160] and so every subcomplex of K is aspherical. Special cases of this are R 'staggered' and a special case of this is any one-relator presentation (p. 161). [See also [C-C-H].]

2. Other situations when $\pi_2 K = 0$ implies $\pi_2 L = 0$ for $L \subset K$ are listed by J.M. Cohen in [Co]. [See also [Hul].] He also states two conjectures which would (together) imply the Whitehead conjecture. [But these conjectures have been shown false in [Bel] and [Du5].]

1 and 2 seem to indicate that a counter example to the Whitehead conjecture would have to be quite complicated, but that *could* be a false impression.