CHAPTER I

R-ALGEBROIDS

0. INTRODUCTION :

We begin this chapter by defining R-algebroids and their morphisms . These have been studied in several papers , [Po-1] , [Mi-1] , [Mi-2] , [Mi-3] , [A-1] .

For instance B.Mitchell [Mi-1,2,3] has given a categorical definition of R-algebroids , and obtained some interesting results on these gadgets . His definition is the following .

Let R be a commutative ring . An R-category A is a category equipped with an R-module structure on each hom set such that composition is R-bilinear . An R-functor is a functor T: A $\dashrightarrow$ B between R-categories such that the maps

$$T : A(a_1,a_2) \dashrightarrow B(Ta_1,Ta_2)$$

are R-linear .

In the language of enriched categories , one can define an R-category to be a category which is enriched over the closed category of R-modules . An R-category with one object is an associative R-algebra with identity .

An R-algebroid A is a small R-category . If A and A' are R-algebroids , define $A\otimes_R A'$ by $Ob(A\otimes_R A') = ObA \times ObA'$,

$$A\otimes_R A'((a,b),(a',b')) = A(a,a') \otimes A'(b,b') .$$

Composition is the unique R-bilinear map satisfying

$$(a\otimes a')(b\otimes b') = ab \otimes a'b' .$$

The <u>enveloping R-algebroid</u> of an R-algebroid A is

$$A^e = A \otimes_R A^{op} \quad .$$

An R-algebroid A is <u>separable</u> if A considered as its own hom functor is projective as an A^e-module . It is <u>central</u> if the map $R \dashrightarrow Hom_{A^e}(A,A)$ is an isomorphism .

Two R-algebroids are <u>Morita equivalent</u> if their module categories are R-equivalent .

Before we state the first reslut of [Mi-3] , let us give the definition of the Brauer group of the commutative ring .

Let R be a commutative ring and let V(R) denote the isomorphism classes of all algebras having R as center and which are separable over R . Let $V_0(R)$ be the subset of V(R) consisting of the algebras $Hom_R(E,E)$ where E is any finitely generated projective faithful R-module . One can prove that V(R) , $V_0(R)$ are closed under the operation of tensor product over R (see [A-G-1]) .

Define an equivalence relation in V(R) as follows : if S_1, S_2 are in V(R) , then S_1 is equvalent to S_2 if there are algebras Δ_1 and Δ_2 in $V_0(R)$ such that $S_1 \otimes_R \Delta_1 \cong S_2 \otimes_R \Delta_2$. Let B(R) denote the set of equivalence classes of V(R) . Then B(R) is an abelian group [A-G-1] .

Now we are ready to state the result given in [Mi-3] ; namely that the Morita class of an R-algebroid A is an element of B(R) if and only if A is central , separable and equivalent to an algebra .

One of the reasons to generalise algebras to algebroids is that an R-algebroid A which is only separable need not be equivalent to an algebra . Thus algebroids give a new direction in the theory of separability .

All the above material has been given in [Mi-1,2,3] .

In [Po-1] , T.Porter has defined an R-algebroid in a slightly different setting . He has defined an R-algebroid A on a fixed set of "object" A_0 to be a disjoint family of R-modules , so that A need not have identities . Also he defined an action of an R-algebroid on a "C-structure" . Finally he defined a crossed module and linked crossed modules with internal groupoids . More precisely , he proved that in the category of R-algebroids over a fixed set , any internal category is an internnal groupoid .

Now we move from this setting to say that it is well known that groups are appropriately generalised to groupoids , (see for example [Br-1],[Hi-1]) . As explained above algebras are appropriately generalised to algebroids ; we give an example in section 1 to illustrate this . Moreover we give the definition of a tensor product between two R-algebroids and reprove the known fact that the category of algebroids is a monoidal closed category [Mi-1] .

In sections 2 and 3 we give the definition of a crossed module over an associative algebra (see for example [Ge-1] , [K-L-1] , [El-1]) and introduce the notion of crossed module over an algebroid . Also we give some properties similar to those well known for crossed modules over groups .

1. R-ALGEBROIDS :

The material of this section may be found in [Mi-1] , [Mi-2] , [Po-1] . We shall give the definition of an

R-algebroid A on a set of "objects" A_0 in the following way :

Recall that A is called a <u>directed graph</u> over a set A_0 if there are given functions ∂^0 , $\partial^1 : A \to A_0$, $\varepsilon : A_0 \to A$, called respectively the <u>source</u> , <u>target</u> and <u>unit</u> maps , such that $\partial^0\varepsilon = \partial^1\varepsilon = 1_{A_0}$. Then we write

$A(x,y) = \{a \in A : \partial^0 a = x , \partial^1 a = y\}$, and write 1_x for εx .

If $a \in A(x,y)$, we also write $a : x \to y$.

An <u>R-algebroid</u> $(A, A_0, \partial^0, \partial^1, \varepsilon, +, .)$ (which is abbreviated to A) is a directed graph A over A_0 together with for all $x,y,z \in A_0$;

i) an R-module structure on each $A(x,y)$,

ii) an R-bilinear function , called composition ,

$$* : A(x,y) \times A(y,z) \to A(x,z) .$$
$$(a , b) \longrightarrow a * b$$

The only axioms are that composition is associative , and that the elements 1_x , $x \in A_0$, act as identities for composition : if $a : x \to y$, then $1_x * a = a * 1_y = a$.

Thus the composition makes A into a small category .

A <u>morphism</u> $f : A \to B$ of R-algebroids A , B is a functor of the underlying categories which is also R-linear on each $A(x,y) \to B(fx,fy)$. The set of all morphisms $A \to B$ is written $\mathrm{Hom}_R(A,B)$. Note that a morphism $f : A \to B$ preserves the identities .

The zero of $A(x,y)$ is written 0 , or 0_{xy} if additional clarity is required . As usual , bilinearity implies $a * 0 = 0$, $0 * a = 0$, whenever these are defined .

<u>Examples</u>:

1) If A_0 has exactly one object , then an R-algebroid over A_0 is an R-algebra .

2) If A is an R-algebroid over A_0 and $x \in A_0$, then $A(x,x)$ is an R-algebra .

We now come to one of the most important features of the category of R-algebroids namely that it has an internal hom functor.

Let A,B be R-algebroids . Suppose given $f,g \in \mathrm{Hom}_R(A,B)$; we define $\underline{\mathrm{Hom}}(f,g)$ to be the set of all "natural transformations " $f \to g$, that is , the set of all functions $b : A_0 \to B$ such that $bx \in B(fx,gx)$, $x \in A_0$, and for all $x,y \in A_0$ and $a \in A(x,y)$ the following square

$$\begin{tikzcd} fx \arrow[r, dashed, "bx"] \arrow[d, "fa"'] & gx \arrow[d, "ga"] \\ fy \arrow[r, dashed, "by"'] & gy \end{tikzcd}$$

commutes . Then $\underline{\mathrm{Hom}}(f,g)$ is given the structure of R-module by $(rb + r'b')x = rbx + r'b'x$, whenever $x \in A_0$ and $r,r' \in R$. There is a bilinear composition

$$\underline{\mathrm{Hom}}(f,g) \times \underline{\mathrm{Hom}}(g,h) \dashrightarrow \underline{\mathrm{Hom}}(f,h)$$
$$(b , b') \dashrightarrow b * b'$$

where $(b * b')x = bx * b'x$. Then we get ;

<u>Proposition 1.1.1</u>: With the above structure , the family

$$\underline{\mathrm{Hom}}(A,B) = \{\underline{\mathrm{Hom}}(f,g)\}_{f,g \in \mathrm{Hom}_R(A,B)}$$

is an R-algebroid . $\square$

A special case is when A,B are R-algebras ; we still get an R-algebroid $\underline{\mathrm{Hom}}(A,B)$ and this is one of the motivating examples for considering the extension from R-algebras to R-algebroids .

<u>Definition 1.1.2</u>: If A,B are two R-algebroids over A_0,B_0 respectively , we define the <u>tensor product</u> $A \otimes_R B$ over $A_0 \times B_0$

to be the family of R-modules

$$\{A(x,y)\otimes_R B(u,v) \ : \ x,y \in A_o \ , \ u,v \in B_o \}$$

with composition $(a'\otimes b') * (a \otimes b) = (a' * a) \otimes (b' * b)$.

<u>Lemma 1.1.3</u>: Let A,B be R-algebroids over A_o, B_o respectively. Then $A\otimes_R B$ is an R-algebroid over $A_o \times B_o$. □

<u>Proposition 1.1.4</u>: Let A,B,C be R-algebroids . Then there is a natural isomorphism between $\mathrm{Hom}_R(A\otimes_R B \ , \ C)$ and $\mathrm{Hom}_R(A,\underline{\mathrm{Hom}}(B,C))$.

<u>Proof</u>:

Define a map $\eta : \mathrm{Hom}_R(A\otimes_R B,C) \to \mathrm{Hom}_R(A,\underline{\mathrm{Hom}}(B,C))$ as follows : if $\Phi : A\otimes_R B \to C$, then $\eta(\Phi) : A \to \underline{\mathrm{Hom}}(B,C)$ and if $x \in Ob(A)$, then $\eta(\Phi)(x)$ is to be a morphism $B \to C$, given on objects by $y \dashrightarrow \Phi(x,y)$ and on arrow $b : y \to y'$ by $(\eta(\Phi)(x))(b) = \Phi(1_x \otimes b)$. If a is an arrow in A , then $\eta(\Phi)(a) \in \underline{\mathrm{Hom}}(B,C)$ which is given on objects by $y \to \Phi(a\otimes 1_y)$ and on arrows $b : y \to y'$ by $(\eta(\Phi)(a))(b) = \Phi(a\otimes b)$.

Define a map $\eta' : \mathrm{Hom}_R(A,\underline{\mathrm{Hom}}(B,C)) \dashrightarrow \mathrm{Hom}_R(A\otimes_R B,C)$ as follows :

if $\Psi : A \dashrightarrow \underline{\mathrm{Hom}}(B,C)$, then $\eta'(\Psi) : A\otimes_R B \dashrightarrow C$. If (x,y) is an object in $Ob(A)\times Ob(B)$, then we define $\eta'(\Psi)(x,y) = \Psi(x)(y)$ and if $a\otimes b$ is an arrow in $A\otimes_R B$ such that $a : x \to x'$, $b : y \to y'$, then $\Psi(x),\Psi(x') : B \to C$ and so $\Psi(a) : \Psi(x) \to \Psi(x')$ and $\Psi(x)(b) : \eta'\Psi(x,y) \to \eta'\Psi(x,y')$, $\Psi(x')(b) : \eta'\Psi(x',y) \to \eta'\Psi(x',y')$.

Thus we get the diagram

$$\begin{tikzcd}
\eta'(\Psi)(x,y) \arrow[r, "\Psi(x)(b)"] \arrow[d, "\Psi(a)(y)"'] & \eta'(\Psi)(x,y') \arrow[d, "\Psi(a)(y')"] \\
\eta'(\Psi)(x',y) \arrow[r, "\Psi(x')(b)"'] & \eta'(\Psi)(x',y')
\end{tikzcd} .$$

Define $\eta'\Psi(a\otimes b) = \Psi(a)(y')\Psi(x)(b) = \Psi(x')(b)\Psi(a)(y)$.

Now we want to show that $\eta\eta' = 1$, $\eta'\eta = 1$. For $\eta\eta' = 1$, let $\Psi : A \dashrightarrow \underline{\mathrm{Hom}}(B,C)$ and let $(x,y) \in Ob(A)\times Ob(B)$, then $\eta\eta'(\Psi)(x,y) = \eta(\Psi(x)(y)) = \Psi(x,y)$. If $a\otimes b \in A\otimes_R B$, then $\eta\eta'(\Psi)(a\otimes b) = \eta(\Psi(a)(y')\Psi(x)(b)) = \Psi(a\otimes 1_y)\Psi(1_x\otimes b) = \Psi(a\otimes b)$. Thus $\eta\eta' = 1$.

For $\eta'\eta = 1$, let $\Phi : A\otimes_R B \dashrightarrow C$ and let $x \in Ob(A)$, then $(\eta'(\eta(\Phi))(x))(y) = \eta'(\Phi(x,y)) = \Phi(x)(y)$ for $y \in Ob(B)$ and $\eta'((\eta(\Phi))(x))(b) = \eta'(\Phi(1_x\otimes b)) = \Phi(1_x)(y')\Phi(x)(b) = \Phi(x)(b)$ for $b:y \to y' \in B$.

Hence $\eta'((\eta(\Phi))(y)) = \eta'(\Phi(a\otimes 1_y)) =$ $\Phi(a)(y)\ \Phi(x)(1_y) = \Phi(a)(b)$, whenever $a : x \to x'$ and $b : y \to y'$. That is , the category of R-algebroids can be given the structure of a monoidal closed category . □

For other properties of the category of R-algebroids which are not valid in the category of R-algebras see , [M-1,2,3] .

If the unit map is omitted from the algebroid structure then we obtain an <u>R-algebroid (without identities)</u> .

<u>Remark 1.1.5</u>: Let A,B be algebroids (without identities) and let $M(A_o,B_o)$ denote the set of functions $A_o \dashrightarrow B_o$. Let θ be the function

$$\theta : \mathrm{Hom}_R(A,B) \longrightarrow M(A_0,B_0) \ .$$
$$f \longrightarrow f_0$$

Then each fibre $\theta^{-1}(h) = \mathrm{Hom}_R(A,B;h)$ can be given the structure of R-module by $(f+g)a = fa + ga$, $(rf)a = f(ra)$, for all $a \in A$, $r \in R$.

2. CROSSED MODULES (OVER ASSOCIATIVE ALGEBRAS):

The general concept of crossed module originates in the work (1949) of J.H.C.Whitehead [Wh-1],[Wh-2] in algebraic topology . There the crossed modules were free crossed modules of groups . Also the notion of crossed module has been studied by Peiffer [Pe-1] and Reidemeister [R-1] , and they have defined identities among relations . For further detail see the survey of Brown-Huebschmann [B-Hu-1] .
In the group case , a crossed module generalises the concepts of a normal subgroup and that of an ordinary module .

The work of [K-L-1] in algebraic K-theory has introduced crossed modules of Lie algebras . In fact they have studied a fibration in Lie-algebras and they found that the induced map of the fibration gives a crossed module . The early work of [Ge-1] , [L-1] and [L-S-1] essentially involves the notion of crossed modules in associative algebras and commutative algebras under different names , which they use to define cohomology groups of algebras . Also [L-R-1] has analysed crossed modules in associative algebras, and the general case of crossed modules in a category of interest C has been discussed in [Po-2] : he has proved that "the category of

internal categories in a category of interest C is equivalent to the category of crossed modules C" . For the precise result , see [Po-2] .

In this section , we give the definition of crossed module in the category of associative algebras in order to set the stage for the definition of crossed module over algebroid in the next section .

Fix a commutative ring R (with unit) , and let <u>AL</u> be the category of associative algebras over R .

We define now an <u>associative action</u> in the category <u>AL</u> as follows :

Let A,M be associative algebras over R . An <u>associative action</u> of A on M is a pair of maps

$$A \times M \to M \quad , \quad M \times A \to M$$

$$(a,m) \to {}^{a}m \quad , \quad (m,a) \to m^{a}$$

such that M is a left and right A-module (bi-A-module) , that is ,

i) $(m+m')^{a} = m^{a} + m'^{a}$, ${}^{a}(m+m') = {}^{a}m + {}^{a}m'$,

ii) $m^{a+a'} = m^{a} + m^{a'}$, ${}^{a+a'}m = {}^{a}m + {}^{a'}m$,

and satisfy the conditions :

iii) $(m.m')^{a} = m \,.\, m'^{a}$, ${}^{a}(m.m') = {}^{a}m \,.\, m'$,

iv) $m^{aa'} = (m^{a})^{a'}$, ${}^{aa'}m = {}^{a}({}^{a'}m)$,

v) $r(m^{a}) = m^{ra} = (rm)^{a}$,

for all $r \in R$, $m,m' \in M$, and $a,a' \in A$.

A <u>crossed module</u> in <u>AL</u> is an associative algebra morphism $\mu : M \to A$ with an associative action of A on M such that :

i) $\mu({}^{a}m) = a.(\mu m)$, $\mu(m^{a}) = (\mu m).a$

ii) ${}^{\mu m}m' = m\,m'$, $m^{\mu m'} = m\,m'$

for all $m, m' \in M$ and $a \in A$.

<u>Examples</u>: 1) Let A be an associative R-algebra and let I be a two-sided ideal in A . Let $i: I \dashrightarrow A$ be the inclusion map , then i with action of A on I given by multiplication is a crossed module .

2) Let A,M be associative algebras and let M be a bi A-module . Then the zero map from M to A is a crossed module with the action given by bimodule structure .

Now we move on and in the next section to give the definition of crossed module (over an algebroid) by using the above definition .

3. <u>CROSSED MODULES (OVER ALGEBROIDS)</u>:

In the previous section , we defined a crossed module in the context of associative algebras .In this section we define a crossed module over an algebroid .

Let A_0 be a set and let A,M be two R-algebroids over A_0 , where M need not have identities . Suppose A operates on M on the right and on the left as follows :

Let $m: x \to y \in M$ and $a \in A(w,x)$, $b \in A(y,z)$, then we denote the right action by $m^b \in M(x,z)$, and the left action by ${}^a m \in M(w,y)$ as shown in the diagram below

$$x \xrightarrow{m} y$$

$$w \xrightarrow{a} x \qquad y \xrightarrow{b} z$$

$${}^a m \in M(w,y) \qquad m^b \in M(x,z)$$

left action right action

such that these actions satisfy the following axioms ;

(1.3.1)

$({}^{a}m)^{b} = {}^{a}(m^{b})$ (1.3.1)(i)

$({}^{m}a)^{b} = {}^{m}ab$, $b({}^{a}m) = b{}^{a}m$ (1.3.1)(ii)

${}^{m}a+b = {}^{m}a + {}^{m}b$, ${}^{a+b}m = {}^{a}m + {}^{b}m$
$(m + m_1)^{b} = m^{b} + m_1{}^{b}$, ${}^{a}(m + m_1) = {}^{a}m + {}^{a}m_1$ (1.3.1)(iii)

$(rm)^{b} = r\, m^{b} = m^{rb}$, ${}^{a}(rm) = r\, {}^{a}m = r{}^{a}m$ (1.3.1)(iv)

${}^{1_x}m = m = m^{1_y}$ (1.3.1)(v)

for all $a,b \in A$, $m,m_1 \in M$ and $x,y \in A_0$. Thus we get :

<u>Definition 1.3.2</u>: Let A,M be two R-algebroids over A_0 . A morphism $\mu: M \dashrightarrow A$ is called a <u>crossed module</u> if there are actions of A on M satisfying the above axioms and also the following axioms :

$\mu(m^{b}) = (\mu m)b$, $\mu({}^{a}m) = a(\mu m)$ (1.3.2)(i)

$mm' = {}^{m}\mu m' = \mu m_{m'}$ (1.3.2)(ii)

for all $a,b \in A$, $m,m' \in M$ and both sides are defined .

<u>Definition 1.3.3</u>: A <u>morphism</u> of crossed modules $(\alpha,\beta):(A,M,\mu) \dashrightarrow (A',M',\mu')$ is two algebroid morphisms $\alpha: A \dashrightarrow A'$, $\beta: M \dashrightarrow M'$ such that $\alpha\mu = \mu'\beta$ and $\beta({}^{a}m) = {}^{\alpha a}\beta m$, $\beta(m^{b}) = \beta m^{\alpha b}$, for all $a,b \in A$, $m \in M$ and $\alpha: A \dashrightarrow A'$ is to preserve identities . Thus we have a category $\underline{C}$ of crossed modules (over algebroids) .

To give examples of such crossed modules , we define a subalgebroid and two-sided ideal .

<u>Definition 1.3.4</u>: Let A be an R-algebroid over A_0 . A <u>subalgebroid</u> A' is a disjoint family of R-submodules

$$\{A'(x,y) \subseteq A(x,y)\}_{x,y \in A_0}$$

with units and each R-bilinear function

$$A'(x,y) \times A'(y,z) \dashrightarrow A'(x,z)$$

is the restriction of the R-bilinear function

$$A(x,y) \times A(y,z) \dashrightarrow A(x,z) .$$

For example , the family $\{A(x,x)\}_{x \in A_0}$ is a subalgebroid .

<u>Definition 1.3.5</u>: Given an R-algebroid A over A_0 , a <u>two-sided ideal</u> I in A is a family of R-submodules

$$\{I(x,y) \subseteq A(x,y)\}_{x,y \in A_0}$$

such that I satisfies the axiom:

if $a \in I(x,y)$, $b \in A(z,x)$, $c \in A(y,w)$, then $ba \in I(z,y)$ and $ac \in I(x,w)$.

<u>Example</u>: Let A be an R-algebroid over A_0 and suppose I is a two-sided ideal in A . Let $i:I \to A$ be the inclusion morphism and let A operate on I by

(i) $a^c = ac$ (ii) ${}^b a = ba$, for all $a \in I$, $b,c \in A$.

Then $i:I \to A$ is a crossed module . Clearly I is an R-algebroid (without identities) .

<u>Remark 1.3.6</u>: Let $f:A \dashrightarrow B$ be an algebroid morphism , where A,B are defined over the same set A_0 and $Ob(f) = 1_{A_0}$. Then $\ker f = \{a \in A(x,y): fa = 0_{xy} \text{ for all } x,y \in A_0\}$ is a two-sided ideal in A .

<u>Proposition 1.3.7</u>: Let $\mu : M \to A$ be a crossed module of algebroids . Then $\mathrm{Im}\,\mu = \{\mu m: m \in M\}$ is a two-sided ideal in A .

<u>Proof</u>: Let $a \in \mathrm{Im}\,\mu$, so there is $m \in M$ such that $\mu m = a$, for some $a \in A$. Let $b \in A$ such that ab is defined , then $ab = \mu m\, b = \mu(m^b)$. Thus $ab \in \mathrm{Im}\,\mu$ and similarly $ca \in \mathrm{Im}\,\mu$, for $c \in A$ and ca is defined . □

Let I be a two-sided ideal in A . Then we can define quotient R-modules $A(x,y)/I(x,y)$ for all $x,y \in A_0$. Then there is an R-bilinear morphism

$*: A(x,y)/I(x,y) \times A(y,z)/I(y,z) \dashrightarrow A(x,z)/I(x,z)$

and associativity holds .

Then we get an R-algebroid A/I which is the family of quotient R-modules

$$\{A(x,y)/I(x,y) : x,y \in A_0\} .$$

We call it the <u>quotient R-algebroid</u> and then there is a canonical mapping $p: A \to A/I$ of R-algebroids . Also we have an exact sequence

$$I \xrightarrow{i} A \xrightarrow{p} A/I .$$

Thus for any crossed module (A,M,μ) , there is an exact sequence $\ker \mu \dashrightarrow A \dashrightarrow \operatorname{Im} \mu$.

We can state some properties of algebroids .

i) Since $\operatorname{Im}\mu$ is a two-sided ideal , then coker $\mu = A/\mu M$ exist and hence there is an exact sequence

$$\operatorname{Im}\mu \dashrightarrow M \dashrightarrow \operatorname{coker} \mu .$$

ii) Since $mm' = {}^{\mu m}m'$, and if $\mu m = 0$, then $m.M = 0$ and $M.m = 0$. Thus $m \in \operatorname{Ann}(M)$ (Ann means annihilator) and clearly Ann(M) is a subalgebroid of M . In particular $\ker\mu.\ker \mu = 0$.

iii) Coker $\mu = A/\operatorname{Im}\mu$ acts on ker μ .

iv) Let $0 \dashrightarrow K \dashrightarrow M \xrightarrow{p} A \dashrightarrow 0$ be a central extention , that is , it is a short exact sequence such that if $k \in K$ and $m \in M$, then $km = mk = 0$. Then $p: M \dashrightarrow A$ can be give the structure of a crossed module .

<u>Proof</u>: For any $a \in A$, let sa denote an element of M such that $p\, sa = a$ (thus s is a section of p) .

Define actions of A on M as follows:

$^{a}m = (sa)m \quad , \quad m^{b} = m(sb)$.

First , to show that these actions are well-defined .

Let s' be a section of p and let $m_1 \in M$, then we want to prove that

$^{a}m = (sa)m = (s'a)m$ for $a \in A$.

Let $m_1 \in M$, then $pm_1 = a$ and hence $m_1 - sa$, $m_1 - s'a \in K$.

So $(m_1 - sa)\, m = (m_1 - s'a)\, m = 0$, then $(sa)m = (s'a)m$. So the left action is well-defined . We can prove similarly that the right action is well-defined . It is clear that these actions satisfy the axioms for a crossed module . □

CHAPTER II

DOUBLE R-ALGEBROIDS

0. INTRODUCTION:

We begin this chapter by showing how to mimic the idea given in chapter I in one higher dimension . That is , we look for "algebroids in two dimensions" . So we need two different additions and compositions .

In fact , we make an analogy to the idea given by R.Brown "Higher dimensional group theory" [Br-2] to define double R-algebroids .

In section 2 we prove that there exist two functors from the category of R-double algebroids to the category of crossed modules . Also we give examples of double R-algebroids in the third section .

1. DEFINITIONS:

The notion of double category has occured often in the literature (see for example , [Be-1],[Gr-1],[Ma-1],[Wy-1], [K-S-1],[B-S-1],[S-W-1] and is due originally to Ehresmann [Eh-1]) . In this section we study an object with more structure than a double category , which we call a double R-algebroid .

To define double R-algebroids , we start to give in some detail the definition of double category ;

Definition 2.1.1:[Eh-1],[B-S-1] By a double category D is meant four related categories

$$(D, D_1, \partial_1^0, \partial_1^1, *_1, c_1) \ , \ (D, D_2, \partial_2^0, \partial_2^1, *_2, c_2)$$

$$(D_1, D_0, \varepsilon_1^0, \varepsilon_1^1, *, c) \ , \ (D_2, D_0, \varepsilon_2^0, \varepsilon_2^1, *, c)$$

as partially shown in the diagram

$$\begin{tikzcd}
 & D \arrow[dl, shift right, "\partial_1^1"'] \arrow[dr, shift right, "\partial_2^1"'] & \\
D_1 \arrow[ur, bend left, "c_1"] \arrow[dr, shift right, "\varepsilon_1^1"'] & & D_2 \arrow[ul, bend right, "c_2"'] \arrow[dl, shift right, "\varepsilon_2^1"'] \\
 & D_0 \arrow[ul, shift right, "\varepsilon_1^0"'] \arrow[ur, shift right, "\varepsilon_2^0"'] \arrow[ul, bend left, "c"] \arrow[ur, bend right, "c"'] &
\end{tikzcd}$$

and satisfying the rules (i-v) given below . The elements of D will be called <u>squares</u> , of D_1, D_2 <u>horizontal</u> and <u>vertical</u> edges respectively , and of D_0 <u>points</u> or <u>objects</u> . We will assume the relation :

i) $\varepsilon_2^i \, \partial_2^j = \varepsilon_1^j \, \partial_1^i \qquad\qquad i, j = 0, 1$

and this allows us to represent a square $\alpha \in D$ as having boundary edges pictured as

$$\begin{tikzcd}
 & \partial_1^0\alpha \arrow[d, dash] & \\
\partial_2^0\alpha \arrow[r, dash] & \alpha \arrow[l, dash] \arrow[r, dash] & \partial_2^1\alpha \arrow[l, dash] \\
 & \partial_1^1\alpha \arrow[u, dash] &
\end{tikzcd}$$

while the edges are pictured as

$$\varepsilon_1^0 a \xleftarrow{\ a\ } \varepsilon_1^1 a \qquad\qquad \begin{tikzcd} \varepsilon_2^0 b \arrow[d, dash, "b"] \\ \varepsilon_2^1 b \end{tikzcd}$$

$a \in D_1 \qquad\qquad b \in D_2$

From now on we will write the boundary of a square as $\underline{\partial}$(the square) for example the boundary of α is written as

$$\underline{\partial}\alpha = \left(\partial_2^0\,\alpha \begin{matrix} \partial_1^0\alpha \\ \\ \partial_1^1\alpha \end{matrix} \partial_2^1\alpha\right) .$$

ii) $\partial_2^i(\epsilon_1 a) = \epsilon\delta_1^i a \qquad i = 0,1$

$\partial_1^j(\epsilon_2 b) = \epsilon\delta_2^j b \qquad j = 0,1$.

So the identities $\epsilon_1 a$, $\epsilon_2 b$ form squares which have boundaries

$$\underline{\partial}(\epsilon_1 a) = \left(\epsilon x \begin{matrix} a \\ \\ a \end{matrix} \epsilon y\right) , \quad \underline{\partial}(\epsilon_2 b) = \left(b \begin{matrix} \epsilon z \\ \\ \epsilon w \end{matrix} b\right) .$$

iii) $\epsilon_1\epsilon x = \epsilon_2\epsilon x$

iv) $\partial_2^i(\alpha *_1 \beta) = \partial_2^i\alpha * \partial_2^i\beta \qquad i = 0,1$

$\partial_1^j(\alpha *_2 \beta) = \partial_1^j\alpha * \partial_1^j\beta \qquad j = 0,1$

for all $\alpha,\beta \in D$ such that both sides are defined .

v) (The interchange law)

$$(\alpha *_1 \beta) *_2 (\gamma *_1 \delta) = (\alpha *_2 \gamma) *_1 (\beta *_2 \delta) ,$$

whenever $\alpha,\beta,\gamma,\delta \in D$ and both sides are defined .

<u>Definition 2.1.2</u>: A <u>double R-algebroid</u> D is four related R-algebroids

$$(D,D_1,\partial_1^0,\ \partial_1^1,\epsilon_1,+_1,*_1,\cdot_1) \ , \ (D,D_2,\partial_2^0,\partial_2^1,\epsilon_2,+_2,*_2,\cdot_2)$$

$$(D_1,D_0,\delta_1^0,\delta_1^1,\epsilon,+,*,.) \quad , \quad (D\ ,D\ ,\delta_2^0,\delta_2^1,\epsilon,+,*,.)$$

as shown in the diagram

$$\begin{tikzcd}
 & D \arrow[dl, shift right, "\partial_1^0"'] \arrow[dr, shift left, "\partial_2^0"] & \\
D_1 \arrow[ur, shift right, "\partial_1^1"'] \arrow[dr, shift left, "\delta_1^0"] & & D_2 \arrow[ul, shift left, "\partial_2^1"] \arrow[dl, shift right, "\delta_2^0"'] \\
 & D_0 \arrow[ul, shift left, "\delta_1^1"] \arrow[ur, shift right, "\delta_2^1"'] &
\end{tikzcd}$$

ϵ_1, ϵ_2, ϵ, ϵ

and satisfying the rules given below .

The elements of D will be called <u>squares</u> , of D_1, D_2 <u>horizontal</u> and <u>vertical</u> edges respectively and of D_0 the set of "objects" .

(2.1.3)

$$\varepsilon_2^i \, \partial_2^j = \varepsilon_1^j \, \partial_1^i \qquad i,j \in \{0,1\} .$$

Then we can represent a square α as having boundary edges given by

$$\begin{tikzcd}[column sep=small] & \partial_1^0\alpha \arrow[d, dash] & \\ \partial_2^0\alpha \arrow[r, dash] & \alpha \arrow[r, dash] & \partial_2^1\alpha \\ & \partial_1^1\alpha \arrow[u, dash] & \end{tikzcd} \qquad \begin{tikzcd} & 2 \\ 1 \arrow[ur] & \end{tikzcd}$$

where the edges pictured as

$$\varepsilon_1^0 a \xleftarrow{\;a\;} \varepsilon_1^1 a \qquad\qquad \begin{tikzcd} \varepsilon_2^0 b \arrow[d, dash, "b"] \\ \varepsilon_2^1 b \end{tikzcd}$$

$$a \in D_1 \qquad\qquad b \in D_2$$

First , we assume on D four operations $+_1, *_1, +_2, *_2$ defined in the following way :

Let $\alpha, \beta, \gamma, \delta, \zeta \in D$ have boundaries given by

$$\underline{\partial}\alpha = (a \overset{c}{\underset{b}{}} d) \;,\; \underline{\partial}\beta = (a_1 \overset{c}{\underset{b}{}} d_1) \;,\; \underline{\partial}\gamma = (a' \overset{b}{\underset{e}{}} d') \;,\; \underline{\partial}\delta = (a \overset{c_1}{\underset{b_1}{}} d)$$

and $\underline{\partial}\zeta = (d \overset{c'}{\underset{b'}{}} e)$.

Then $\alpha +_1 \beta$, $\alpha *_1 \gamma$, $\alpha +_2 \delta$, $\alpha *_2 \zeta$ have boundary edges in the form

$$\underline{\partial}(\alpha +_1 \beta) = (a+a_1 \overset{c}{\underset{b}{}} d+d_1) \;,\; \underline{\partial}(\alpha *_1 \gamma) = (aa' \overset{c}{\underset{e}{}} dd') \;,$$

$$\underline{\partial}(\alpha +_2 \zeta) = (a \begin{smallmatrix} c+c_1 \\ b+b_1 \end{smallmatrix} d) \ , \ \underline{\partial}(\alpha *_2 \zeta) = (a \begin{smallmatrix} cc' \\ bb' \end{smallmatrix} e) \ .$$

So we are ready to give more rules for double R-algebroid

(2.1.4)

$$\partial_2^i(\alpha +_1 \beta) = \partial_2^i\alpha + \partial_2^i\beta \qquad i = 0,1 \quad (2.1.4)(i)$$

$$\partial_1^i(\alpha +_2 \beta) = \partial_1^i\alpha + \partial_1^i\beta \qquad i = 0,1 \quad (2.1.4)(ii)$$

$$\partial_2^i(\alpha *_1 \beta) = \partial_2^i\alpha * \partial_2^i\beta \qquad i = 0,1 \quad (2.1.4)(iii)$$

$$\partial_1^i(\alpha *_2 \beta) = \partial_1^i\alpha * \partial_1^i\beta \qquad i = 0,1 \quad (2.1.4)(iv)$$

for all $\alpha, \beta \in D$ and both sides are defined .

(2.1.5)

We have two scalar multiplications ; for $\alpha \in D$ as above and $r \in R$, so we define $r \cdot_1 \alpha$, $r \cdot_2 \alpha$ to have boundary edges in the form

$$\underline{\partial}(r \cdot_1 \alpha) = (ra \begin{smallmatrix} c \\ b \end{smallmatrix} rd) \ , \ \underline{\partial}(r \cdot_2 \alpha) = (a \begin{smallmatrix} rc \\ rb \end{smallmatrix} d) \ .$$

These multiplications are to satisfy the following axioms :

$$\left.\begin{array}{l} r \cdot_1 (\alpha +_2 \beta) = (r \cdot_1 \alpha) +_2 (r \cdot_1 \beta) \\ r \cdot_2 (\alpha +_1 \beta) = (r \cdot_2 \alpha) +_1 (r \cdot_2 \beta) \end{array}\right\} \qquad (2.1.5)(i)$$

$$\left.\begin{array}{l} r \cdot_1 (\alpha *_2 \beta) = (r \cdot_1 \alpha) *_2 (r \cdot_1 \beta) \\ r \cdot_2 (\alpha *_1 \beta) = (r \cdot_2 \alpha) *_1 (r \cdot_2 \beta) \end{array}\right\} \qquad (2.1.5)(ii)$$

$$r \cdot_1 (s \cdot_2 \alpha) = s \cdot_2 (r \cdot_1 \alpha) \qquad (2.1.5)(iii)$$

for all $\alpha, \beta \in D$, $r, s \in R$ and both sides are defined .

These rules make sense in terms of boundaries , for example , let $\alpha, \beta \in D$ have boundaries given by

$$\underline{\partial}\alpha = (a \begin{smallmatrix} c \\ b \end{smallmatrix} d) \ , \ \underline{\partial}\beta = (a \begin{smallmatrix} c_1 \\ b_1 \end{smallmatrix} d) \ , \text{ then } \underline{\partial}(r \cdot_1 \alpha) = (ra \begin{smallmatrix} c \\ b \end{smallmatrix} rd) \ ,$$

$$\underline{\partial}(r \cdot_1 \beta) = (ra \begin{smallmatrix} c_1 \\ b_1 \end{smallmatrix} rd) \ , \ \underline{\partial}[r \cdot_1 (\alpha +_2 \beta)] = (ra \begin{smallmatrix} c+c_1 \\ b+b_1 \end{smallmatrix} rd) \ ,$$

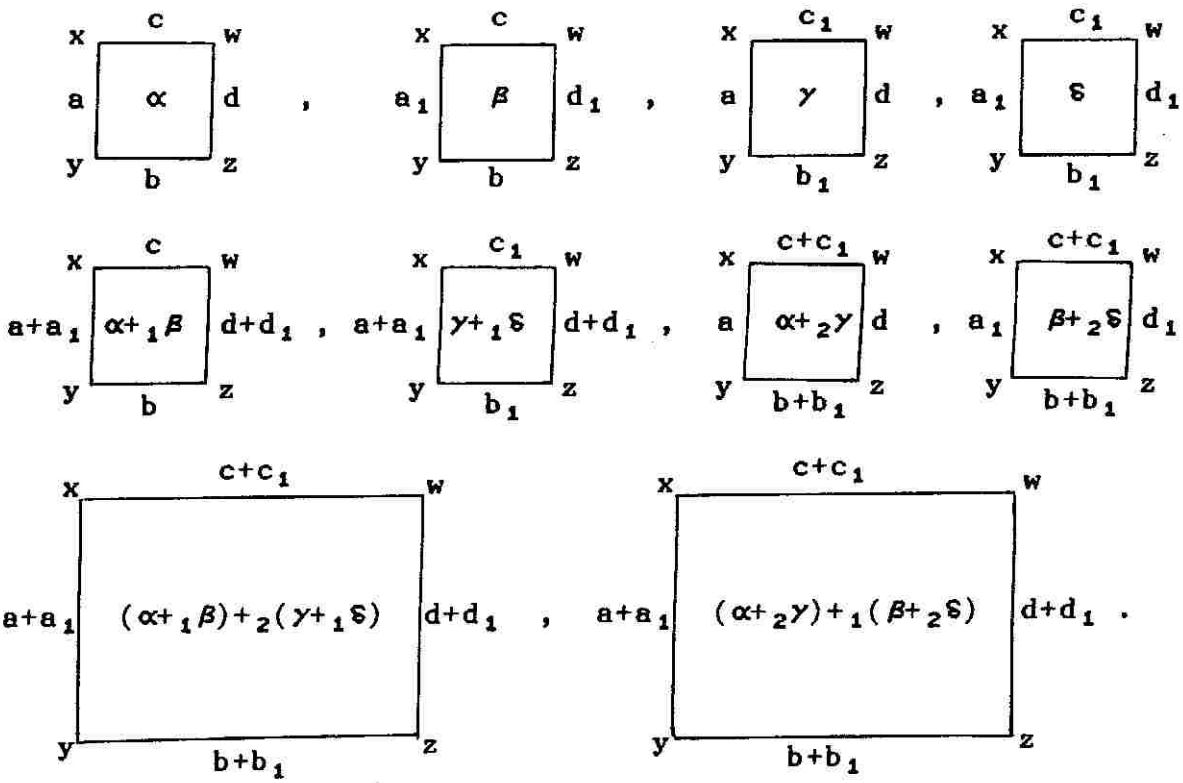
and $\underline{\partial}[(r \cdot_1 \alpha) +_2 (r \cdot_1 \beta)] = (ra \begin{smallmatrix} c+c_1 \\ b+b_1 \end{smallmatrix} rd)$, that is ,

$r \cdot_1 (\alpha +_2 \beta) = (r \cdot_1 \alpha) +_2 (r \cdot_1 \beta)$ in terms of boundaries .

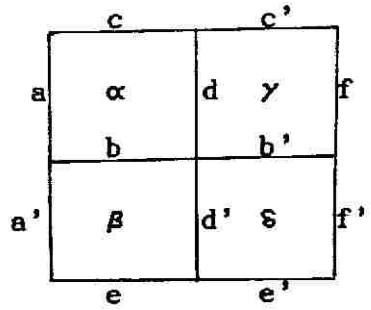
(2.1.6) (The interchange laws) :

$(\alpha +_1 \beta) +_2 (\gamma +_1 \delta) = (\alpha +_2 \gamma) +_1 (\beta +_2 \delta)$ (2.1.6)(i)

which is diagrammatically as shown below :



$(\alpha *_1 \beta) *_2 (\gamma *_1 \delta) = (\alpha *_2 \gamma) *_1 (\beta *_2 \delta)$ (2.1.6)(ii)



$(\alpha +_1 \beta) *_2 (\gamma +_1 \delta) = (\alpha *_2 \gamma) +_1 (\beta *_2 \delta)$ (2.1.6)(iii)

which is diagrammatically given by

$$\begin{tikzcd} x \arrow[r, "cc'"] & u \\ y \arrow[u, "a+a_1"] \arrow[r, "bb'"'] & v \arrow[u, "f+f_1"'] \end{tikzcd} \quad (\alpha +_1 \beta) *_2 (\gamma +_1 \delta) \quad , \quad \begin{tikzcd} x \arrow[r, "cc'"] & u \\ y \arrow[u, "a+a_1"] \arrow[r, "bb'"'] & v \arrow[u, "f+f_1"'] \end{tikzcd} \quad (\alpha *_2 \gamma) +_1 (\beta *_2 \delta) \ .$$

$(\alpha +_2 \beta) *_1 (\gamma +_2 \delta) = (\alpha *_1 \gamma) +_2 (\beta *_1 \delta)$ (2.1.6)(iv) .

The explanation is similar to that for the interchange law (iii) , whenever $\alpha, \beta, \gamma, \delta \in D$ and both sides are defined .

(2.1.7)

We assume that each of the algebroid structures has identities and then ε_1 , ε_2 give these identities in the following way ;

given $a \in D_1(x,y)$, $b \in D_2(x,y)$, then $\varepsilon_1 a$, $\varepsilon_2 b$ having boundaries given by ;

$\underline{\partial}(\varepsilon_1 a) = (1_x \overset{a}{\underset{a}{\ }} 1_y)$, $\underline{\partial}(\varepsilon_2 b) = (b \overset{1_x}{\underset{1_y}{\ }} b)$, such that ε_1 , ε_2 are algebroid morphisms and satisfy the following axiom ;

let $a, a_1 \in D_1(x,y)$ and $b, b_1 \in D_2(x,y)$, then

$\varepsilon_1(a+a_1) = \varepsilon_1 a +_2 \varepsilon_1 a_1$, $\varepsilon_2(b+b_1) = \varepsilon_2 b +_1 \varepsilon_2 b_1$.

We shall need later some simple facts on zero elements namely ;

<u>Remark 2.1.8</u>: If $x, y \in D_0$, then we write 0 or 0_{xy} for the zero elements in both $D_1(x,y)$ and $D_2(x,y)$. However if $c \in D_1(x,y)$, $b \in D_1(z,w)$, then we have a set $D^1(c,b) = (\partial_1^0)^{-1}(c) \cap (\partial_1^1)^{-1}(b)$ and this set has zero which we write 0^1_{cb} . The boundaries of this element are given by

$$\begin{array}{ccc} x & c & y \\ 0 & 0^1_{cb} & 0 \\ z & b & w \end{array}$$

and it is clear that 0^1_{cb} is the zero for $+_1$ in $D^1(c,b)$, where $D^1(c,b)$ is the set of arrows in direction 1 , from c to b .

Also we can get a square 0^2_{ad} with boundaries given in the form

$$\begin{array}{ccc} x & 0 & z \\ a & 0^2_{ad} & d \\ y & 0 & w \end{array} \quad ,$$

which is the zero for $+_2$ in $D^2(a,d)$. Notice that , if $\alpha \in D$ is given by

$$\begin{array}{ccc} & b & \\ d & \alpha & f \\ & e & \end{array} \quad ,$$

then $0^1_{cb} *_1 \alpha = 0^1_{ce}$, $0^2_{ad} *_2 \alpha = 0^2_{af}$ by distributivity .

<u>Definition 2.1.9</u>: A <u>morphism</u> between two double R-algebroids D , E (over the same set of objects) is a triple of algebroid morphisms

$\Psi_0 : D \dashrightarrow E$, $\Psi_1 : D_1 \dashrightarrow E_1$, $\Psi_2 : D_2 \dashrightarrow E2$

which preserves all structures . Thus we get a category of double R-algebroids . Also we can define a morphism between two double R-algebroids on different sets of objects , by using the definition given in chapter 1 section 1 . Let us denote the category of double R-algebroids (over the same set of objects) by <u>DA</u> .

2. FUNCTORS (DOUBLE ALGEBROIDS) $\longrightarrow$ (CROSSED MODULES):

In chapter 1 section 2 and in the previous section , we have defined two categories namely the category of crossed modules $\underline{C}$ and the category of double R-algebroids $\underline{DA}$.

In this section , we make an analogy with the result given in ([B-S-1] , proposition 1) that is , we want to show how to obtain from a double R-algebroid two crossed modules (over algebroids) . We start with the main result of this section namely ;

<u>Proposition 2.2.1</u>: If D is a double R-algebroid , then we have two crossed modules associated with D .

<u>Proof</u>: First , let $A_0 = D_0$ (the set of objects of D) , and $A_2 = D_1$, the algebroid of arrows of D_1 . We take M_2 to consist of squares β with boundary of the form $\left(1 \begin{smallmatrix} m \\ 0 \end{smallmatrix} 1\right)$,

that is ,

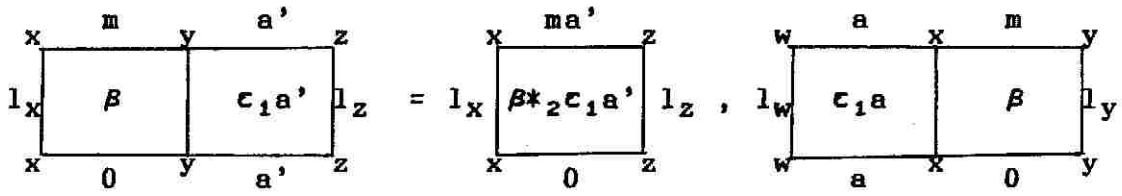
$$M_2(x,y) = \{\beta\in D: \partial_1^0\beta = m \ , \ \partial_1^1\beta = 0_{xy} \ , \ \partial_2^0\beta = 1_x, \ \partial_2^1\beta = 1_y\} \ .$$

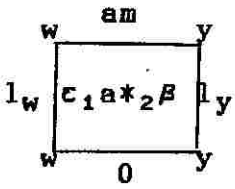
We define $+,*,.$ on M_2 by $\beta + \beta_1 = \beta +_2 \beta_1$, $\beta * \beta' = \beta *_2 \beta'$ and $r \ . \ \beta = r \ ._2 \ \beta$, whenever $\beta, \beta_1, \beta' \in M_2$ and $r \in R$. Thus M_2 is an R-algebroid over A_0 . Let $\beta \in M_2$ as above and let $a' \in A_2(y,z)$, $a \in A_2(w,x)$. So we get two squares in the form

$$\begin{tikzcd} y \arrow[r, "a'"] \arrow[d, "1_y"'] & z \arrow[d, "1_z"] \\ y \arrow[r, "a'"'] & z \end{tikzcd} \ \beta \ , \qquad \begin{tikzcd} w \arrow[r, "a"] \arrow[d, "1_w"'] & x \arrow[d, "1_x"] \\ w \arrow[r, "a"'] & x \end{tikzcd} \ \beta \ .$$

Then we define the right and the left actions of A_2 on M_2 by the formulae

$\beta a' = \beta *_2 \epsilon_1 a'$, $a\beta = \epsilon_1 a *_2 \beta$ as shown below :





We now prove that these actions satisfy the axioms for crossed modules (1.3.1)(i-iv) .

Axioms (1.3.1)(i-ii) , follow directly from the associativity of $*_2$.

(1.3.1)(iii)

$\beta^{a+b} = \beta^a + \beta^b$, ${}^{a+b}\beta = {}^a\beta + {}^b\beta$

$(\beta + \beta_1)^b = \beta^b + \beta_1{}^b$, ${}^a(\beta + \beta_1) = {}^a\beta + {}^a\beta_1$

<u>Proof</u>:

$$\begin{aligned} \beta^{a+b} &= \beta *_2 \epsilon_1(a + b) && \text{by definition} \\ &= \beta *_2 [\epsilon_1 a +_2 \epsilon_1 b] && \text{by (2.1.7)(i)} \\ &= (\beta *_2 \epsilon_1 a) +_2 (\beta *_2 \epsilon_1 b) && \text{by distributivity} \\ &= \beta^a + \beta^b . \end{aligned}$$

We prove similarly that ${}^{a+b}\beta = {}^a\beta + {}^b\beta$, $(\beta + \beta_1) = \beta^b + \beta_1{}^b$, ${}^a(\beta + \beta_1) = {}^a\beta + {}^a\beta_1$.

For (1.3.1)(iv) , namely $(r.\beta)^a = r . \beta^a = \beta^{ra}$ for all $r \in R$,

$$\begin{aligned} (r.\beta)^a &= (r ._2 \beta) *_2 \epsilon_1 a && \text{by definition} \\ &= r ._2 (\beta *_2 \epsilon_1 a) && \text{from bilinearity} \\ &= r ._2 \beta^a = r . \beta^a . \end{aligned}$$

Also by the definition and bilinearity , we get $(r.\beta)^a = \beta^{ra}$.

Clearly (1.3.1)(iv) is satisfied .

Define now a map $\mu_2 : M_2 \dashrightarrow A_2$ by $\mu_2\beta = \partial_1^0\beta$

It is clear that μ_2 is an algebroid morphism .

Finally , to prove that (A_2, M_2, μ_2) is a crossed module , it suffices to verify the axioms (1.3.2)(i-ii) ; namely

$\mu_2(\beta^a) = (\mu_2\beta)a$, $\mu_2({}^a\beta) = a\,(\mu_2\beta)$, $\beta\,\beta' = \beta^{\mu_2\beta'} = {}^{\mu_2\beta}\beta'$.

The first part is clear . Thus we just want to show that $\beta\,\beta' = \beta^{\mu_2\beta'} = {}^{\mu_2\beta}\beta'$.

Suppose β , β' have boundaries in the form $(1\ {}^{m}_{0}\ 1)$, $(1\ {}^{m'}_{0}\ 1)$. Then

$\beta * \beta' = \beta *_2 \beta'$ by definition

$= (\epsilon_1 m *_1 \beta) *_2 (\beta' *_1 \epsilon_1 0)$ by the identity rule

$= (\epsilon_1 m *_2 \beta') *_1 (\beta *_2 \epsilon_1 0)$ by (2.1.6)(ii) .

Since $\beta *_2 \epsilon_1 0 = \epsilon_1 0$ by remark (2.1.8) , we have

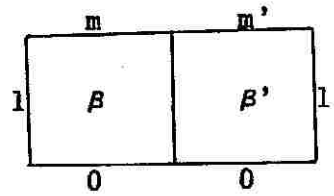
$\beta * \beta' = (\epsilon_1 m *_2 \beta') *_2 \epsilon_1 0 = \epsilon_1 m *_2 \beta' = {}^{m}\beta' = {}^{\mu_2\beta}\beta'$

by the definition and remark (2.1.8) .

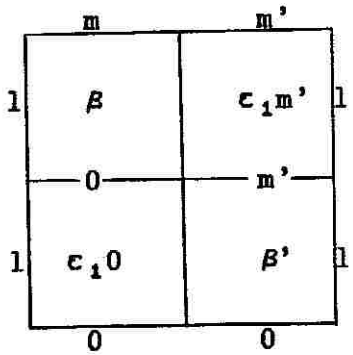
We can use similar argument to get

$\beta * \beta' = \beta^{\mu_2\beta'}$, as shown in the diagram below

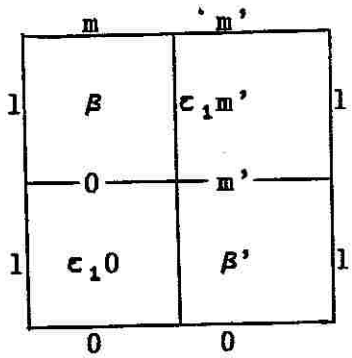
$\beta * \beta' = \beta *_2 \beta' =$



=



=



=

$$= \begin{array}{ccc} & mm' & \\ 1 & \boxed{\beta^{m'}} & 1 \\ & 0 & \end{array}$$

$= \beta^{m'} = \beta^{\mu_1 \beta'}$.

Then we get a crossed module (A_2, M_2, μ_2) .

For the second crossed module , we assume $A_1 = D_2$ and take M_1 to consist of squares β with boundary of the form $(m \begin{smallmatrix} 1 \\ 1 \end{smallmatrix} 0)$, that is ,

$M_1(x,y) = \{\beta \in D : \partial_2^0 \beta = m \ , \ \partial_2^1 \beta = 0_{xy} \ , \partial_1^0 \beta = 1_x \ , \ \partial_1^1 \beta = 1_y\}$

and clearly M_1 is an R-algebroid over A_0 by $\beta + \beta_1 = \beta +_1 \beta_1$, $\beta * \beta' = \beta *_1 \beta'$ and $r \cdot \beta = r \cdot_1 \beta$. Then we can use similar argument as above to get a crossed module (A_1, M_1, μ_1) . This is the complete proof of the proposition . □

The next section gives examples of double R-algebroids .

3. EXAMPLES:

We give in this section three examples of double algebroids .

1) Let B be an R-algebroid over B_0 . Then we can construct a double R-algebroid $D = \square B$ of commuting squares in B such that D and B have the same set of objects (i.e. $D_0 = B_0$) .

Let $D_1 = D_2 = B$ be the horizontal and vertical algebroid structures , and let D consist of quadruples $\alpha =$ $(a \begin{smallmatrix} c \\ b \end{smallmatrix} d)$ for $a,b,c,d \in B$ and $cd = ab$.

Thus α is determined by its boundary edges .

We define now $+_1$, $+_2$, $*_1$, $*_2$, $\cdot_1$, $\cdot_2$ on D in the following way :

Let $\alpha = (a \begin{smallmatrix} c \\ b \end{smallmatrix} d)$, $\beta = (a_1 \begin{smallmatrix} c \\ b \end{smallmatrix} d_1)$, $\gamma = (a \begin{smallmatrix} c_1 \\ b_1 \end{smallmatrix} d)$,

$\delta = (a' \begin{smallmatrix} b \\ e \end{smallmatrix} d')$, $\zeta = (d \begin{smallmatrix} c' \\ b' \end{smallmatrix} e)$, then we define

$\alpha +_1 \beta = (a+a' \begin{smallmatrix} c \\ b \end{smallmatrix} d+d')$, $\alpha +_2 \gamma = (a \begin{smallmatrix} c+c_1 \\ b+b_1 \end{smallmatrix} d)$,

$\alpha *_1 \delta = (aa' \begin{smallmatrix} c \\ e \end{smallmatrix} dd')$, $\alpha *_2 \zeta = (a \begin{smallmatrix} cc' \\ bb' \end{smallmatrix} e)$. If $r \in R$,

we define $r \cdot_1 \alpha = (ra \begin{smallmatrix} c \\ b \end{smallmatrix} rd)$ and $r \cdot_2 \alpha = (a \begin{smallmatrix} rc \\ rb \end{smallmatrix} d)$.

It is clear that these operations are well-defined , for example $+_1$, since $\alpha, \beta \in D$, then $ab = cd$ and $a_1 b = cd_1$ hence $(a + a_1)b = c(d + d_1)$, so $\alpha +_1 \beta \in D$.

Now we want to show that this structure satisfies the axioms for double algebroids .

It is obvious that this structure satisfies the axioms (2.1.2)(i-iv) , (2.1.3)(i-iii) and (2.1.5) .

Thus it is enough to satisfy the axioms (2.1.4)(i-iv) ,

for (2.1.4)(i) , let $\alpha, \beta, \gamma, \delta \in D$ having boundaries given by

$$\alpha = (a \begin{smallmatrix} c \\ b \end{smallmatrix} d) , \beta = (a_1 \begin{smallmatrix} c \\ b \end{smallmatrix} d_1) , \gamma = (a \begin{smallmatrix} c_1 \\ b_1 \end{smallmatrix} d) , \delta = (a_1 \begin{smallmatrix} c_1 \\ b_1 \end{smallmatrix} d_1) ,$$

so

$$\alpha +_1 \beta = (a+a_1 \begin{smallmatrix} c \\ b \end{smallmatrix} d+d_1) , \gamma +_1 \delta = (a+a_1 \begin{smallmatrix} c_1 \\ b_1 \end{smallmatrix} d+d_1) ,$$

$$\alpha +_2 \gamma = (a \begin{smallmatrix} c+c_1 \\ b+b_1 \end{smallmatrix} d) \quad \text{and} \quad \beta +_2 \delta = (a_1 \begin{smallmatrix} c+c_1 \\ b+b_1 \end{smallmatrix} d_1) \text{ and then}$$

$$(\alpha +_1 \beta) +_2 (\gamma +_1 \delta) = (a+a_1 \begin{smallmatrix} c+c_1 \\ b+b_1 \end{smallmatrix} d+d_1) , \text{ and}$$

$(\alpha +_2 \gamma) +_1 (\beta +_2 \delta) = (a+a_1 \begin{smallmatrix} c+c_1 \\ b+b_1 \end{smallmatrix} d+d_1)$.

So $(a+a_1)(b+b_1) = (c+c_1)(d+d_1)$ (since $ab = cd$, $a_1b = cd_1$, $ab_1 = c_1d$, and $a_1b_1 = c_1d_1$. The explanation for (2.1.4)(ii-iv) is similar to that of (2.1.4)(i) .

Thus the structure $\square B$ with these operations does satisfy the rules for a double algebroid . If B contains identities , then $\square B$ contains identities .

2) Let B be an R-algebra and let B_1 , B_2 be two subalgebras of B . Define $D = \square(B_1,B_2)$ to be the set of commuting squares

$\alpha = (a \begin{smallmatrix} c \\ b \end{smallmatrix} d)$, for $a,d \in B_1$, $c,b \in B_2$ and $ab = cd$. Let $D_0 = \{*\}$. If we define the operations $+_1$, $+_2$, $*_1$, $*_2$, $\cdot_1$, $\cdot_2$ on D in a similar way to that in example (1) , we get a double R-algebroid .

3) A generalisation of example (2) is: if B is an R-algebra and B_1 , B_2 are subalgebras of B and given homomorphisms $\Phi : B_1 \to B$, $\Psi : B_2 \to B$.

Define now , $D_0 = \{*\}$ and $D_1 = B_1$, $D_2 = B_2$ and D to consist of quadruples $(a \begin{smallmatrix} c \\ b \end{smallmatrix} d)$, for $a,d \in B_1$, $c,d \in B_2$ such that $(\Phi a)(\Psi b) = (\Psi c)(\Phi d)$.

We define $+_1$, $+_2$, $*_1$, $*_2$, $\cdot_1$, $\cdot_2$ on D in the following way :

for $+_1$, let $\alpha = (a \begin{smallmatrix} c \\ b \end{smallmatrix} d)$, $\beta = (a_1 \begin{smallmatrix} c \\ b \end{smallmatrix} d_1)$, then

$\alpha +_1 \beta = (a+a_1 \begin{smallmatrix} c \\ b \end{smallmatrix} d+d_1)$. So we want to show that

$(\Phi(a+a_1))(\Psi b) = (\Psi c)(\Phi(d+d_1))$, and this equation follows from these two equations $(\Phi a)(\Psi b) = (\Psi c)(\Phi d)$ and $(\Phi a_1)(\Psi b) = (\Psi c)(\Phi d_1)$ and Φ is a morphism .

For $+_2$, $*_1$, $*_2$, $\cdot_1$, $\cdot_2$, we can define these operations similar as in example (2) by using the fact that Φ , Ψ are algebra morphisms .

Clearly the above structure does satisfy the axioms of a double R-algebroid . Moreover , the two associated crossed modules of the above double algebroid are :

i) the first crossed module is given by the morphism

$$B_2 \xrightarrow{I} B_2$$

$$\begin{pmatrix} 1 & c \\ & 0 \end{pmatrix} \longrightarrow c \quad .$$

ii) The second crossed module is given by the morphism

$$B_1 \xrightarrow{I} B_1$$

$$\begin{pmatrix} a & 1 \\ & 1 \end{pmatrix} \longrightarrow a \quad .$$