The Seifert-van Kampen Theorem for the fundamental groupoid of a space with a set of base points

This note is an extract for the convenience of readers of a section of the book [BHS11], with some additional comments.

1 Proof of the Seifert–van Kampen Theorem (groupoid case)

In this section we give the full proof that the morphism of groupoids induced by inclusions

$$\eta \colon \pi_1(X_1, A_1) *_{\pi_1(X_0,A_0)} \pi_1(X_2, A_2) \to \pi_1(X, A) \qquad (1)$$

is an isomorphism when X_1, X_2 are open subsets of $X = X_1 \cup X_2$ and A meets each path component of X_1, X_2 and $X_0 = X_1 \cap X_2$. Here we write $A_\lambda = X_\lambda \cap A$ for $\lambda = 1, 2, 12$.

What one would expect is that the proof would construct directly an inverse to η. Alternatively, the proof would verify in turn that η is surjective and injective.

The proof we give might at first seem roundabout, but in fact it follows the important procedure of *verifying a universal property.* One advantage of this procedure is that we do not need to show that the free product with amalgamation of groupoids exists in general, nor do we need to give a construction of it at this stage. Instead we define the free product with amalgamation by its universal property, which enables us to go directly to an efficient proof of the Seifert–van Kampen Theorem. It also turns out that the universal property guides many explicit calculations. More importantly, the proof guides other results, such the higher dimensional ones in this book.

We use the notion of pushout. Here is the definition for groupoids. We say that the groupoid G and the two morphisms of groupoids $b_1 \colon G_1 \to G$ and $b_2 \colon G_2 \to G$ are the *pushout* of the two morphisms of groupoids $a_1 \colon G \to G_1$ and $a_2 \colon G \to G_2$ if they satisfy the two axioms:

Pushout 1) the diagram

$$\begin{array}{ccc} G_0 & \xrightarrow{a_1} & G_1 \\ {\scriptstyle a_2}\downarrow & & \downarrow{\scriptstyle b_1} \\ G_2 & \xrightarrow[b_2]{} & G \end{array}$$

is a commutative square, i.e. $b_1a_1 = b_2a_2$,

Pushout 2) the previous diagram is universal with respect to this type of diagram, i.e. for any groupoid K and morphisms of groupoids $k_1 \colon G_1 \to K$ and $k_2 \colon G_2 \to K$ such that the following diagram is commutative

$$\begin{array}{ccc} G_0 & \xrightarrow{a_1} & G_1 \\ {\scriptstyle a_2}\downarrow & & \downarrow{\scriptstyle k_1} \\ G_2 & \xrightarrow[k_2]{} & K \end{array}$$

there is a unique morphism of groupoids $k \colon G \to K$ such that $kb_1 = k_1, kb_2 = k_2$. The two diagrams are often combined into one as follows:

$$\begin{array}{cccc} G_0 & \xrightarrow{a_1} & G_1 & \\ {\scriptstyle a_2}\downarrow & & \downarrow{\scriptstyle b_1} & \searrow{\scriptstyle k_1} \\ G_2 & \xrightarrow{b_2} & G & \\ & {\scriptstyle k_2}\searrow & & \overset{k}{\searrow} \; K. \end{array}$$

We think of a pushout square as given by a standard input, the pair (a_1, a_2), and a standard output, the pair (b_1, b_2). The properties of this standard output are defined by reference to *all other* commutative squares with the same (a_1, a_2). At first sight this might seem strange, and logically invalid. However a pushout square is somewhat like a computer program: given the data of another commutative square of the right type, then the output will be a morphism (k in the above diagram) with certain defined properties.

It is a basic feature of universal properties that the standard output, in this case the pair (b_1, b_2) making the diagram commute, is determined up to isomorphism by the standard input (a_1, a_2).

We now state and prove the Seifert–van Kampen theorem for the fundamental groupoid on a set of base points in the case of a cover by two open sets. The reason for giving this in detail is that the proofs of the analogous theorems in higher dimensions are modelled on this one, but need new gadgets of higher homotopy groupoids to realise them.[1]

Theorem 1.1 *Let X_1, X_2 be open subsets of X whose union is X and let A be a subset of $X_0 = X_1 \cap X_2$ meeting each path component of X_1, X_2, X_0 (and therefore of X). Let $A_i = X_i \cap A$ for $i = 1, 2, 0$. Then the following diagram of morphisms induced by inclusion*

$$\begin{array}{ccc} \pi_1(X_0, A_0) & \xrightarrow{a_1} & \pi_1(X_1, A_1) \\ {\scriptstyle a_2}\downarrow & & \downarrow{\scriptstyle b_1} \\ \pi_1(X_2, A_2) & \xrightarrow[b_2]{} & \pi_1(X, A) \end{array}$$

is a pushout of groupoids.

Proof We suppose given a commutative diagram of morphisms of groupoids

$$\begin{array}{ccc} \pi_1(X_0, A_0) & \xrightarrow{a_1} & \pi_1(X_1, A_1) \\ {\scriptstyle a_2}\downarrow & & \downarrow{\scriptstyle k_1} \\ \pi_1(X_2, A_2) & \xrightarrow[k_2]{} & K \end{array}$$

We have to prove that there is a unique morphism $k\colon \pi_1(X, A) \to K$ such that $kb_1 = k_1, kb_2 = k_2$.

We write $b_{12}\colon \pi_1(X_0, A_0) \to \pi_1(X, A)$ for the composite $b_1a_1 = b_2a_2$, write $k_{12} = k_1b_1 = k_2b_2$, and also write b_i for the map of spaces $X_i \to X$.

Let us take an element $[\alpha] \in \pi_1(X, A)$ with representative $\alpha\colon (I, \partial I) \to (X, A)$. Suppose first α has image in X_λ for $\lambda = 1$ or 2. Then $\alpha = a_\lambda\beta$ for $\beta\colon (I, \partial I) \to (X_\lambda, A_\lambda)$ and we define $k[\alpha] = k_\lambda[\beta]$. The condition $k_1a_1 = k_2a_2$ ensures this definition is independent of the choice of λ if α has image in $X_1 \cap X_2$, but it still has to be shown the definition is independent of the choice of α in its class.

We now consider a general $[\alpha]$. By the Lebesgue Covering Lemma ([Bro06, 3.6.4]) there is a subdivision

$$0 = t_0 < t_1 < \cdots < t_{n-1} < t_n = 1$$

of I into intervals by equidistant points such that α maps each $[t_i, t_{i+1}]$ into X_1 or X_2 (possibly in both). Choose one of these written X^i for each i. The subdivision determines a decomposition

$$\alpha = \alpha_0\alpha_1 \dots \alpha_{n-1}$$

such that α_i has image in X^i. Of course the point $\alpha(t_i)$ need not lie in A, but it lies in $X^i \cap X^{i-1}$ and this intersection may be X_1, X_2 or X_0. By the connectivity conditions, for each $i = 0, 1, \cdots, n-1$, we may choose a path γ_i in $X^i \cap X^{i-1}$ joining $\alpha(t_i)$ to A. Moreover, if $\alpha(t_i)$ already lies in A we choose γ_i to be the constant path at $\alpha(t_i)$. In particular γ_0 and γ_n are constant paths. The following figure shows the path α in black and the paths γ_i in white:

Now for each $0 \leqslant i < n$ the path $\beta_i = \gamma_i^{-1}\alpha_i\gamma_{i+1}$ lies in X^i and joins points of A. Notice that β_i also represents a class in $\pi_1(X^i, A)$, which maps by b^i (which may be b_1, b_2, b_{12}) to $\pi_1(X, A)$. It is clear that

$$[\alpha] = b^0[\beta_0]b^1[\beta_1] \dots b^{n-1}[\beta_{n-1}] \in \pi_1(X, A).$$

If there exists the homomorphism k of groupoids that makes the external square commute then the value of $k([\alpha])$ is determined by the above subdivision as

$$\begin{aligned} k([\alpha]) &= k(b^0[\beta_0]b^1[\beta_1] \dots b^{n-1}[\beta_{n-1}]) \\ &= k^0[\beta_0]k^1[\beta_1] \dots k^{n-1}[\beta_{n-1}]. \end{aligned}$$

This proves uniqueness of k, and also proves that $\pi_1(X, A)$ is generated as a groupoid by the images of $\pi_1(X_1, A_1), \pi_1(X_2, A_2)$ by b_1, b_2 respectively.

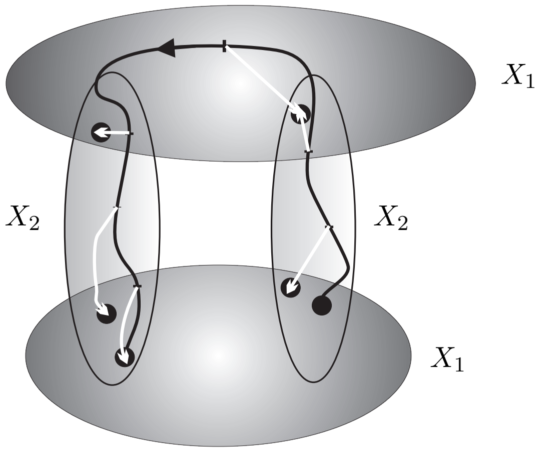
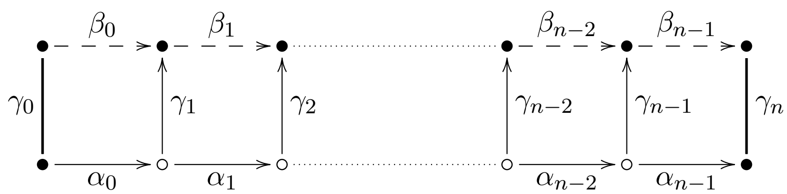


Figure 1: A decomposition of a path α in a Seifert–van Kampen type situation

We have yet to prove that the element $k([\alpha])$ is independent of all the choices made. Before going into that, notice that the construction we have just made can be interpreted diagrammatically as follows. The starting situation looks like the bottom side of the diagram



(2)

where the solid circles denote points which definitely lie in A, and in which γ_0, γ_n are constant paths. The path β_i may be obtained from the other three paths in its square by composing with a retraction from above, as shown in Fig. 2.

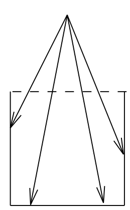


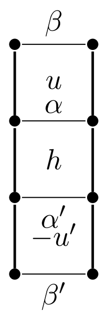
Figure 2: Retraction from above-centre

This retraction also provides a homotopy

$$u\colon \alpha \simeq \beta = (b^0\beta_0)(b^1\beta_1)\dots(b^{n-1}\beta_{n-1}) \tag{3}$$

rel end points. This is the first of many possible and useful filling arguments where we define a map on parts of the boundary of a cube and extend the map to the whole cube using appropriate retractions. [2]

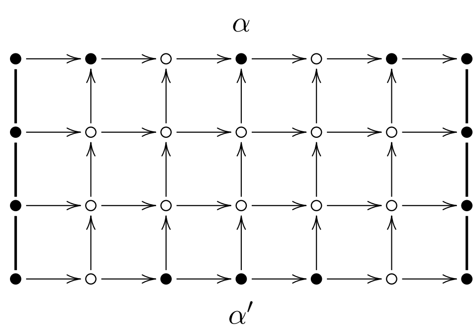
We shall use another filling argument in I^3 to prove independence of choices. Suppose that we have a homotopy rel end points $h\colon \alpha \simeq \alpha'$ of two maps $(I, \partial I) \to (X, A)$. We can perform the construction of a homotopy in (3) for each of α, α', and then glue the three homotopies together. Here thick lines denote constant paths.



(4)

So, replacing βs by αs, we can assume the maps α, α' have subdivisions $\alpha = [\alpha_i]$, $\alpha' = [\alpha'_j]$ such that each α_i, α'_j has end points in A and has image in one of X_1, X_2. Since h is a map $I^2 \to X$, we may again by the Lebesgue covering lemma make a subdivision $h = [h_{lm}]$ such that each h_{lm} lies in one of X_1, X_2. Also by further subdivision as necessary, we may assume this subdivision of h refines on $I \times \partial I$ the given subdivisions of α, α'.

The problem is that none of the vertices of this subdivision are necessarily mapped into A, except those on $\partial I \times I$ (since the homotopy is rel vertices and α, α' both map ∂I to A) and those on $I \times \partial I$ determined by the initial subdivisions of α, α'. So the situation looks like the following:



(5)

Again thick lines denote constant paths. We want to deform the homotopy h to a new homotopy $\bar{h}\colon \bar{\alpha} \simeq \bar{\alpha}'$ again rel end points such that:

$[\alpha] = [\bar{\alpha}], [\alpha'] = [\bar{\alpha}']$ in $\pi_1(X, A)$;

h' has the same subdivision as does h;

any subsquare mapped by h into X_i, $i = 1, 2, 12$ remains so in h';

and any vertex already in A is not moved.

This deformation is constructed inductively on dimension of cells of the subdivision by what we call 'filling arguments' in the cube I^3.

Let us imagine the 3-dimensional cube I^3 as $I^2 \times I$ where I^2 has the subdivision we are working with in h. Define the bottom map to be h. We have to fill I^3 so that in the top face we get a similar diagram but with all the vertices solid, i.e. in A, and each subsquare in the top face lies in the same X_i as the corresponding in the bottom one.

We start by defining the deformation on all 'vertical' edges $\{v\} \times I$ arising from vertices v in the partition of I^2. If the image of a vertex lies in A, then v is to be deformed by a constant deformation; otherwise, we consider the 4,2, or 1 squares of which v is a vertex, let X^v be the intersection of the sets of the cover into which these are mapped, and choose a path in X^v joining $h(v)$ to a point of A. Let us write e_{lm} for the path we have chosen between the vertex $h(s_l, t_m)$ and A. (These e_{lm} are constant if $h(s_l, t_m)$ lies already in A). This gives us the map on the vertical edges of I^3 as in Figure 3.

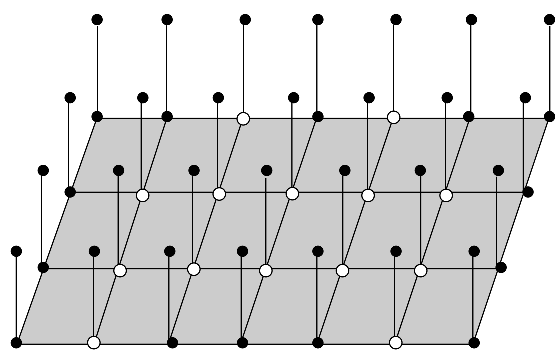


Figure 3: Extending to the edges

From now on, we restrict our construction to the part of I^3 over the square $S_{lm} = [s_l, s_{l+1}] \times [t_m, t_{m+1}]$ and fix some notation for the restriction of h to its sides, $\sigma_{lm} = h|_{[s_l, s_{l+1}] \times \{t_m\}}$ and $\tau_{lm} = h|_{\{s_l\} \times [t_m, t_{m+1}]}$. Then, using the retraction of Figure 2 on each lateral face, we can fill all the faces of a 3-cube except the top one. Now, using the retraction from a point on a line perpendicular to the centre of the top face, as in the following Figure 4

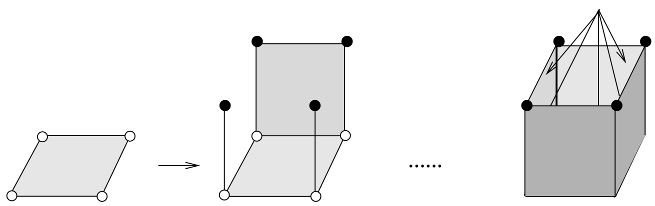
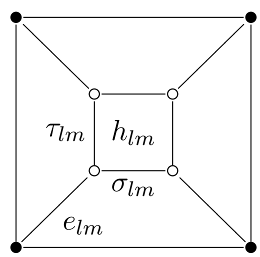


Figure 4: Extending to the lateral faces

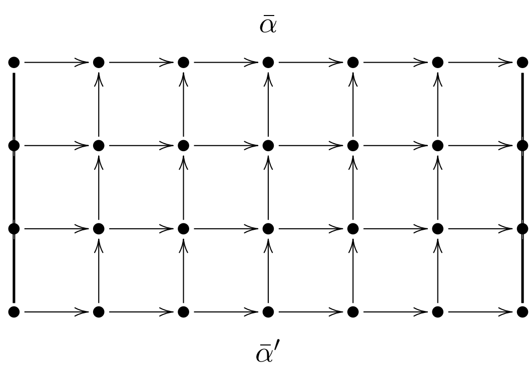
we get at the top face a map that looks like



(6)

and in particular is a map into X^i sending all vertices into A.

If we do the above construction in each square of the subdivision, we get a top face of the cube that is an homotopy $\bar{h}$ rel end points between two paths in the same classes as α and α', and subdivided in such a way that each subsquare goes into some X_λ and sends all vertices of the subsquare into A. Each of these squares produces a commutative square say σ_{ij} of path classes in one of $\pi_1(X_\lambda, A)$, $\lambda = 1, 2$. Thus the diagram can be pictured as



Applying the appropriate k_λ to a subsquare σ_{ij} we get a commutative square l_{ij} in K. Since $k_1a_1 = k_2a_2$, we get that the l_{ij} compose in K to give a square l in K.

Now comes the vital point. Since *any composite of commutative squares in a groupoid is itself a commutative square*, the composite square l is commutative.

But because of the way we constructed it, two sides of this composite commutative square l in K are identities, as the images of the class of constant paths. Therefore the opposite sides of l are equal. This shows that our element $k([\alpha])$ is independent of the choices made, and so proves that k is well defined as a function on arrows of the fundamental groupoid $\pi_1(X, A)$.

The proof that k is a morphism is now quite simple, while uniqueness has already been shown. So we have shown that the diagram in the statement of the theorem is a pushout of groupoids.

This completes the proof. □

There is another way of expressing the above argument on the composition of commutative squares being a commutative square, namely by working on formulae for each individual square as in the expression $a = cdb^{-1}$ for the first square in the following diagram (7), which shows a composite of two squares.

$$\begin{array}{ccccc} \bullet & \xrightarrow{a} & \bullet & \xrightarrow{e} & \bullet \\ {\scriptstyle c}\downarrow & & {\scriptstyle b}\downarrow & & \downarrow{\scriptstyle f} \\ \bullet & \xrightarrow[d]{} & \bullet & \xrightarrow[g]{} & \bullet \end{array} \tag{7}$$

Calculating for the 'composite' of the two squares allows cancelation of the middle term

$$ae = (cdb^{-1})(bgf^{-1}) = cdgf^{-1}$$

which if $c = 1, f = 1$ reduces to $ae = dg$. This argument extends to longer gluings of commutative squares, and hence extends, by induction, and in the other direction, to a subdivision of a square.

Remark 1.2 Essentially the same proof gives a result for arbitrary covers of X by the interiors of sets $U_\lambda, \lambda \in \Lambda$ but then one needs the notion of coequaliser instead of pushout. Also one needs for the same type of proof the condition that A meets every path component of every 3-fold intersection of the sets U_λ. The details are in [BRS84]. One uses the Lebesgue covering dimension. □

Remark 1.3 The original motivation for generalising this theorem from groups to groupoids was the desire to obtain a theorem from which one could deduce the fundamental group of the circle, a basic example in algebraic topology. There are two other points to be made here. One is that the notion of groupoid for describing for example journeys between stations on a railway system is much more intuitive than the usual notion of 'change of base point', which places the emphasis on return journeys. The other remark is that it is not sensible to use only the notion of group in the situation of a connected space X with a cover by say 2 open sets each with 20 path components and whose intersection has 120 path components. The groupoid Seifert-van Kampen theorem accomplishes the transition from topology to algebra, and then to determine the fundamental group at some chosen point you need a combinatorial analysis of the situation, and is a standard technique in combinatorial group theory. □

Remark 1.4 The fact that one can do the calculations in the case of the last remark, and get a precise answer, is not in the traditions of algebraic topology, which tends to get answers by exact sequences, without a precise answer when two adjacent dimensions interact. This problem is clear from the argument using nonabelian cohomology in [Olu58, Bro65]. Thus the method of using groupoids is apparently more powerful than that using nonabelian cohomology. This advantage continues in higher dimensions. □

Remark 1.5 It was contemplating the above proof in 1965 that suggested to R. Brown that the proof should generalise to all dimensions, or at least to dimension 2, if one had the right homotopical gadgets to express the ideas of (i) algebraic inverse to subdivision, (ii) the notion of commutative cube, and (iii) any composition of commutative cubes was commutative. It took some 9 years to make these ideas begin to work, using the notion first of homotopy double groupoid of a pair of spaces, and then the cubical homotopy ω-groupoid of a filtered space, as detailed in the book [BHS11]. □

References

[Bro65] Brown, R. 'On a method of P. Olum'. *J. London Math. Soc.* **40** (1965) 303–304.

[Bro06] Brown, R. *Topology and Groupoids.* Printed by Booksurge LLC, Charleston, S. Carolina, third edition (2006).

[BHS11] Brown, R., Higgins, P. J. and Sivera, R. *Nonabelian algebraic topology: filtered spaces, crossed complexes, cubical homotopy groupoids.* EMS Tracts in Mathematics Vol 15. www.bangor.ac.uk/r.brown/nonab-a-t.html (2011).

[BRS84] Brown, R. and Razak Salleh, A. 'A van Kampen theorem for unions of nonconnected spaces'. *Arch. Math. (Basel)* **42** (1) (1984) 85–88.

[Cro59] Crowell, R. H. 'On the van Kampen theorem'. *Pacific J. Math.* **9** (1959) 43–50.

[Olu58] Olum, P. 'Non-abelian cohomology and van Kampen's theorem'. *Ann. of Math. (2)* **68** (1958) 658–668.

[Whi41] Whitehead, J. H. C. 'On incidence matrices, nuclei and homotopy types'. *Ann. of Math. (2)* **42** (1941) 1197–1239.

[Whi50] Whitehead, J. H. C. 'Simple homotopy types'. *Amer. J. Math.* **72** (1950) 1–57.

Notes

[1]p. 2 This proof differs from that in [Bro06] in working directly with path classes in $\pi_1(X, A)$ instead of first doing the case $A = X$ and then using a retraction. That retraction argument is not so easy to extend to the case of an arbitrary open cover of X, and, more importantly, seemingly impossible to extend to higher dimensions. So the proof we give returns in essence to the argument in [Cro59]. The result for general covers with best possible connectivity conditions is given in [BRS84].

[2]p. 5 These collapsing techniques were developed by J.H.C. Whitehead in [Whi41, Whi50] and have become an important tool in geometric topology.