## The non abelian tensor product of groups

This is a brief introduction to the bibliography on non abelian tensor products of groups.

The basic idea of the non abelian tensor product is simple.

It was early realised that a direct definition of tensor product of non abelian groups gave nothing new. Let $G, H$ be groups. Define $G \otimes H$ as the group with generators $g \otimes h, g \in G, h \in H$ and relations

$$(gg' \otimes h) = (g \otimes h)(g' \otimes h)$$

$$(g \otimes hh') = (g \otimes h)(g \otimes h')$$

for all $g, g' \in G, h, h' \in H$. Then expanding $gg' \otimes hh'$ in two ways yields (after some cancellation)

$$(g \otimes h')(g' \otimes h) = (g' \otimes h)(g \otimes h').$$

From this one finds easily that

$$G \otimes h = (G^{ab}) \otimes_Z (H^{ab}).$$

The start of a new approach was to recognise that if one is interested in non commutative groups then one is certainly interested in the commutator map $(g, h) \mapsto ghg^{-1}h^{-1}$ on a group $G$

$$[\ ,\ ] : G \times G \to G.$$

This map is not bimultiplicative but instead satisfies

$$[gg', h] = [{}^g g', {}^g h][g, h],$$

$$[g, hh'] = [g, h][{}^h g, {}^h h'],$$

where ${}^h g = hgh^{-1}$. A map of this type we call a *biderivation*. It is thus natural to consider the universal object for biderivations. So we now define $G \otimes G$ to be the group with generators $g \otimes h, g, h \in G$ and relations

$$(gg' \otimes h) = ({}^g g' \otimes {}^g h)(g \otimes h)$$

$$(g \otimes hh') = (g \otimes h)({}^h g \otimes {}^h h')$$

for all $g, g', h, h' \in G$. The natural map $G \times G \to G \otimes G, (g, h) \mapsto g \otimes h$ is then the universal biderivation and any biderivation $b : G \times G \to L$ factors uniquely to give a morphism of groups $b' : G \otimes G \to L$. In particular the commutator map defines a morphism $\kappa : G \otimes G \to G$ whose image of course is the commutator subgroup $[G, G]$ of $G$.

There are other relations satisfied by the commutator, for example $[g, g] = 1$. It is natural therefore to consider another construction, the exterior product

$$G \wedge G = (G \otimes G)/\{g \otimes g\}.$$

Again the commutator map yields a map

$$\kappa' : G \wedge G \to G.$$

This map was introduced in essence in 1952 by Claire Miller, who proved that the kernel of $\kappa'$ is isomorphic to the Schur Multiplicator $H_2(G)$.

The next advance took place in the mid 1970s. Lue published a paper giving a definition of $P \otimes G$ in the case when $G \to P$ is a crossed module. He also gave an interesting adjointness result, which was much later taken up by Guin.

At about this time Dennis released a preprint which developed a theory of the tensor square and related constructions, including

$$(G \wedge^{\sim} G) = (G \otimes G)/\{(g \otimes g')(g' \otimes g)\}.$$

Some of the relations developed in the preprint were useful later to Brown and Loday.

The motivation for Loday for the notion of cat$^n$-group (which he called at first $n$-cat-groups) came from attempts at generalisations of relative K-theory, which he had shown needed the notion of crossed module. For multirelative K-theory it seemed that what was needed was 'crossed modules of crossed modules'. This led to the notion of crossed square and its equivalence with cat$^2$-groups.

A visit by Brown to Strasbourg in 1981 led to a formulation of a GVKT for cat$^n$-groups, and in 1982 in Bangor we saw that a particular pushout of 'degenerate' crossed squares would imply a construction of a tensor product for crossed $P$-modules. This was a key part of the paper in Topology, 1987.

The time taken to write up the work, and then to go through several refereeing steps, led to several papers in this area appearing at nearly the same time in 1987.

A key result proved using homology was that the tensor square of a finite group is finite, and this was extended by Ellis to the general case. This suggested the need for calculations, and then computer calculations, as one part of work by Brown-Johnson-Robertson, and this work continues as an interesting group theory problem in its own right. A complication is that the tensor square of a finitely generated group need not be finitely generated: an easy counterexample is the free group on two generators.

An interesting variant occurred in the mod $q$ tensor product, whose final formulation took some experimentation. This has now proved of interest in generalising work in isologism from groups to $p$-groups.

Work of N. Innassaridze solved the problem of the derived functors of $G \otimes -$, using techniques of non abelian homological algebra. This required giving a new definition of $G \otimes H$ for the case of non compatible actions in order to carry over useful properties to the more general case.

Later work of N. and H. Innassaridze extends work of Guin in using the non abelian tensor product to define new homology groups of a group with coefficients in another group. This homology has applications to K-theory, as shown by Guin and extended by Innassaridze.

Work of Ellis gives a notion of generalised center for a group. An element $g \in G$ is in the center of $G$ if $[g, h] = 1$ for all $h \in G$. There are three other more refined conditions on $g$ namely for all $h \in G$:

$$g \otimes h = 1;$$

$$g \wedge^{\sim} h = 1;$$

$$g \wedge h = 1.$$

All these have interest from the group theory viewpoint.

Conclusion

This story again illustrates how the work aimed at higher dimensional group theory has found links with classical material, which can then be viewed in a new light. It confirms an intuition that the shaky vision of grand prospects is probably a misty view of an undiscovered world.