Interpretations of Yetter's notion of G-coloring : simplicial fibre bundles and non-abelian cohomology

Tim Porter

October 11, 1995

In [20], Yetter makes the following definition:

Fix a finite group, G. For any space X and a triangulation, $\mathbf{T}$, a G-coloring of $\mathbf{T}$ is a map $\lambda : \mathbf{T}_{(1)} \to G$ such that given any $\sigma \in \mathbf{T}_{(2)}$, $\lambda(e_1)^{\varepsilon_1}\lambda(e_2)^{\varepsilon_2}\lambda(e_3)^{\varepsilon_3} = 1$, whenever $\partial\sigma = e_1^{\varepsilon_1}e_2^{\varepsilon_2}e_3^{\varepsilon_3}$, for $\varepsilon_i = \pm 1$ denoting in the first expression non-inversion or inversion in the group G and in the second preservation or reversal of orientation. We denote the set of all G-colorings of $\mathbf{T}$ by $\Lambda_G(\mathbf{T})$.

Yetter then defines $Z_G(X, \mathbf{T})$ to be the vector space having $\Lambda_G(\mathbf{T})$ as basis. Restricting to the case where X is a surface, he shows that if $\mathbf{T}'$ is a triangulation obtained from $\mathbf{T}$ by iterated subdivision of edges, then there is a well defined map $\mathrm{res}_{\mathbf{T}',\mathbf{T}} : \Lambda_G(\mathbf{T}') \to \Lambda_G(\mathbf{T})$ which induces a map $res_{\mathbf{T}',\mathbf{T}}$ on the corresponding vector spaces. The $Z_G(X, \mathbf{T})$s and $res_{\mathbf{T}',\mathbf{T}}$ define a diagram of vector spaces and he takes $Z_G(X)$ to be the colimit of this diagram. It is then shown that

1. this defines a $(2 + 1)$-dimensional topological quantum field theory in the sense of Atiyah [1];
2. the vector space $Z_G(X)$ is isomorphic to the vector space whose basis is the set of conjugacy classes of representations from $\Pi(X)$ to G (i.e. of natural isomorphism classes of functors from $\Pi(X)$ to G, regarded as a groupoid with one object).

Yetter then extends the construction to take coefficients in a crossed G-set and in a second paper, [21], shows how to adapt the method to handling coefficients in an algebraic model of a homotopy 2-type. In both cases the theory gives a TQFT and there are hints at an interpretation in terms analogous to 2. above. Here we will provide alternative proofs of some of Yetter's results. This gives an interpretation in terms of simplicial fibre bundles and of 2-descent data or non-abelian cocycles.

Acknowledgements: I would like to thank my colleagues in the University of Wales Mathematical Physics and Physical Mathematics Video Seminar. This weekly seminar is held over the University videolink. The existence of this link allows mathematicians in the various colleges of the University to attend seminars of interest to the group when attendence in person would be out of the question due to the distances between the Colleges. I would like to thank the organisers of the seminar and the HEFCW for the link that made the seminars possible.

1 Colorings in a simplicial groupoid.

1.1 Background

Yetter's initial idea is summarised above. In [21], he took coefficients in a finite cat^1-group (cf. Loday, [17]) or a categorical group as he chooses to call it. For the moment it is sufficient to consider such a gadget in the form of its nerve in the category direction. This is a simplicial group whose Moore complex has length 1, thus having trivial groups in all dimensions other than 0 and 1. (The finiteness restriction imposed by Yetter will not be needed for the moment.) We denote this simplicial group by G.

Let X be a manifold and $\mathbf{T} = (T, \phi : |T| \to X)$, a triangulation of X. In this context Yetter defines a G-coloring, λ, of $\mathbf{T}$ to be an assignment to each edge T of a 0-simplex of G and to each 2-simplex of T of a 1-simplex in G, such that if $\sigma =< a, b, c >$ is an ordered 2-simplex

$$d_1(\lambda(\sigma)) = \lambda(< a, b >)\lambda(< b, c >)$$

and

$$d_0(\lambda(\sigma)) = \lambda(< a, c >),$$

and the faces of every ordered 3-simplex commute as a diagram in G.

Here the theory is designed for use with a simplicial group whose Moore complex is length 1, so there is no need to define what happens in dimensions greater than 3. Comparison with the definition with coefficients in a group, given earlier, shows that the relation of equality has been replaced by a possibly non-trivial 2-simplex joining the two composites coming from $< a, b, c >$. The commutativity of the 3-simplex then says that nothing new happens in that dimension. If one goes to a simplicial group with a longer Moore complex, there is an obvious way of attempting to generalise this construction. This way would be messy to say the least.

A triangulation gives a simplicial complex. As such it also gives rise to a simplicial set. If T is the simplicial complex, the corresponding simplicial set is obtained by adding in all the degenerate simplices generated by the simplices of T. If, as is often the case here, the simplicial complex T is ordered, so that the vertices of T form an ordered set, then the simplicial set generated by T has as n-simplices the symbols $< v_0, \ldots, v_n >$ where the v_i are vertices of T with $v_0 \leq \ldots \leq v_n$, and in which

$$d_i < v_0, \ldots, v_n >=< v_0, \ldots, \hat{v}_i, \ldots, v_n >,$$

$$s_i < v_0, \ldots, v_n >=< v_0, \ldots, v_i, v_i, \ldots, v_n > .$$

Given any reduced simplicial set, K, that is one in which K_0 consists of one point only, there is a well known construction of Kan that gives a simplicial group, GK, often called the *loop group* of K. This construction will not be given in detail here as we will need a generalisation of it that we will introduce shortly. The classical treatment of it can be found in the survey by Curtis, [10]. That survey is also a good initial source for much of the simplicial set theory that we will be needing in this article, but beware of misprints. A more thorough treatment is given in May, [18]. We will use (standard) notation from [18] wherever possible.

The way found initially around the restriction that K had to be reduced in the above loop construction was to take a maximal tree in K and to contract it to a point. In 1984, a groupoid version of the loop group construction was given by Dwyer and Kan, [12]. (Unfortunately the published paper has many misprints and the cleaned-up version that we will use was prepared by my student Phil Ehlers as part of his master's dissertation, [13]. Alternatives have been proposed by Joyal and Tierney, and by Moerdijk and Svensson. They end up with simplicial objects in the category of groupoids, whilst the Dwyer - Kan version gives a simplicially enriched groupoid, i.e. a groupoid all of whose Hom-objects are simplicial sets. A simplicially enriched groupoid is also a simplicial groupoid (simplicial object in the category of groupoids), but is one whose object of objects is a constant simplicial set.)

Let SS denote the category of simplicial sets and $SGpds$ that of simplicially enriched groupoids or as we will often call them, simply, simplicial groupoids. The loop groupoid functor is a functor

$$G : SS \to SGpds$$

which takes the simplicial set K to the simplicially enriched groupoid GK where $(GK)_n$ is the free groupoid on the graph

$$K_{n+1} \rightrightarrows K_0,$$

where the two functions are $s = (d_1)^{n+1}$ and $t = d_0(d_2)^n$, with relations $s_0 x = id$ for $x \in K_n$. The degeneracy maps are given on generators by $s_i^{GK}(x) = s_{i+1}^K(x)$ for $x \in K_{n+1}$. The face maps are given on generators by $d_i^{GK}(x) = d_{i+1}^K(x)$ for $x \in K_{n+1}$ for $0 < i \leq n$, and $d_0^{GK}(x) = (d_1^K(x))(d_0^K(x))^{-1}$. Note that the groupoid at each level is free.

There is a classifying 'space' functor, $\overline{W} : SGpds \to SS$ that is right adjoint to G. If H is a simplicially enriched groupoid, then $\overline{W}H$ is the simplicial set described by

- $(\overline{W}H)_0 = ob(H_0)$;
- $(\overline{W}H)_1 = arr(H_0)$, the set of arrows of the groupoid H_0;

and for $n \geq 2$,

- $(\overline{W}H)_n = \{(h_{n-1}, \ldots, h_0) | h_i \in arr(H_i) \text{ and } dom(h_{i-1}) = cod(h_i), 0 < i < n\}$.

The face and degeneracy maps between $(\overline{W}H)_1$ and $(\overline{W}H)_0$ are the source and target maps and the identity maps respectively, whilst the face and degeneracy maps at higher levels are given as follows :
$d_0(h_{n-1}, \ldots, h_0) = (h_{n-2}, \ldots h_0)$;
$d_n(h_{n-1}, \ldots, h_0) = (d^H_{n-1} h_{n-1}, \ldots, d^H_1 h_1)$;
and for $0 < i < n$,
$d_i(h_{n-1}, \ldots, h_0) = (d^H_{i-1} h_{i-1}, d^H_{i-2} h_{i-2}, \ldots, d^H_0 h_{n-i} h_{n-i-1}, h_{n-i-2}, \ldots, h_0)$;
whilst
$s_0(h_{n-1}, \ldots, h_0) = (id_{dom(h_{n-1})}, h_{n-1}, \ldots, h_0)$;
and for $0 < i \leq n$,
$s_i(h_{n-1}, \ldots, h_0) = (s^H_{i-1} h_{n-1}, \ldots, s^H_0 h_{n-i}, id_{cod(h_{n-i})}, h_{n-i-1} \ldots, h_0)$.

Remark:

If H is a simplicial *group* then the classifying 'space' $\overline{W}H$ defined in this way reduces to the more classical version given by Kan, which may conveniently be found in Curtis, [10] or May, [18]. We will return to the theory of classifying spaces slightly later on.

It is easier in what follows to work with ordered triangulations although this is not strictly necessary. Thus if $\sigma \in T_{(n)}$, it is of the form $< a_0, a_1, \ldots, a_n >$, where $a_0 < a_1 < \ldots < a_n$ in the ordering of the vertices. Although there is a slight risk with such an abuse of notation, we will also write σ for the corresponding generator in $G(T)_{n-1}$. Such a σ will have source $< a_0 >$ and target $< a_1 >$ as is easily checked.

1.2 The group case

We will begin by examining the case treated by Yetter in the first of the two papers.

Lemma 1.1 *Let* $\mathbf{T}$ *be an ordered triangulation of a space* X. *Let* G *a group and* $K(G,0)$ *the corresponding simplicial group with* G *in all dimensions and with all face and degeneracy maps being the identity map on* G.

Suppose that λ *is a* G*-coloring of* $\mathbf{T}$, *then* λ *defines a simplicial groupoid morphism*

$$\lambda' : G(T) \to K(G,0),$$

given by

$$\lambda'_0 < a, b >= \lambda < a, b > \in G = K(G,0)_0;$$

if $\sigma =< a_0, a_1, \ldots, a_n > \in T_{(n+1)}$,

$$\lambda'_n \sigma = s_0^n \lambda < a_0, a_1 > .$$

Proof.
As we only have to define what λ' does on the non-degenerate simplices of $G(T)$, it suffices to check that the simplicial identities work for this choice of $\lambda'_n\sigma$. The majority of the calculation is without interest, but for $n = 1$,

$$\lambda'_1 < a_0, a_1, a_2 >= s_0\lambda < a_0, a_1 >= s_0\lambda < a_0, a_2 > s_0\lambda < a_1, a_2 >^{-1}$$

since $\lambda < a_0, a_1 > \lambda < a_1, a_2 > \lambda < a_0, a_2 >^{-1}= 1$, so

$$d_0\lambda_1 < a_0, a_1, a_2 >= \lambda < a_0, a_1 >= \lambda_0 < a_0, a_1 >$$

whilst we also have

$$d_0\lambda_1 < a_0, a_1, a_2 >= \lambda < a_0, a_1 >= \lambda < a_0, a_2 > \lambda < a_1, a_2 >^{-1} .$$

Comparing this with

$$\lambda_0 d_0 < a_0, a_1, a_2 >= \lambda_0(< a_0, a_2 >< a_1, a_2 >^{-1}) = \lambda < a_0, a_2 > \lambda < a_1, a_2 >^{-1},$$

we see that λ_1 satisfies the relevant relations for a simplicial map. □

This assignment of simplicial maps to G-colorings is easily seen to be bijective, so the set of G-colorings, $\Lambda_G(T)$, is identifiable with the set of simplicial groupoid maps from $G(T)$ to $K(G,0)$.

This means that a coloring λ can be thought of as being

$$\lambda : G(T) \to K(G,0),$$

but as the Moore complex of $K(G,0)$ is just G in dimension zero, λ factors through the groupoid $\pi_0 G(T)$, which is exactly the edge path groupoid of the polyhedron $|T| \cong X$, i.e. up to equivalence, the fundamental groupoid of X. Thus λ corresponds to a groupoid map $\lambda : \Pi X \to G$. Such a groupoid map is well known to correspond to a covering space of X and as G is assumed to be finite, the covering space will have finite fibre.

Another way to approach this covering space aspect of this simplest class of colorings is to note that, by the adjointness of $G(\ )$ and $\overline{W}$, λ corresponds to a simplicial map $\overline{\lambda} : T \to \overline{W}K(G,0)$. On $\overline{W}K(G,0)$, there is a universal principal G-bundle

$$WK(G,0) \to \overline{W}K(G,0),$$

(cf. May, [18], p.88) and $\overline{\lambda}$ will thus induce a simplicial principal $K(G,0)$-bundle (G-torsor) back on T. As G is finite, this is just a simplicial covering space as before. The bundle (G-torsor) is a twisted cartesian product with λ as its twisting function.

The simplicial nature of these objects arises because of the use of triangulations. They may therefore seem slightly less 'continuous' than might be hoped for. To lessen this impression of dependence on a triangulation, a final interpretation using open covers is useful, although it does not avoid a passage to finer covers. This interpretation is via the use of the Čech cocycle description of G-torsors and ideas from elementary non-abelian cohomology.

Given any (finite) open cover $\mathcal{U} = (\mathcal{U}_\alpha)$ of X, there is a triangulation $\mathbf{T}$ such that the star open cover associated to $\mathbf{T}$ is finer than $\mathcal{U}$. (The details are well known and are easily accessible in 'classical' texts on algebraic topology, e.g. Spanier [19].) Such an observation has the advantage that it makes clear that the triangulation is subsidiary to the construction as one can replace it by the nerve of an open cover of X. In this interpretation a G-coloring of T is just a 1-cocycle with values in G, subordinate to the cover determined by the triangulation, $\mathbf{T}$. (This description <u>does</u> skim over the surface of some difficulties, but these will be addressed later on, and do not influence the end product.)

1.3 When G is a cat^1-group

In the second paper, [21], of the two considered here, Yetter considers colorings in a categorical group, i.e. a small category endowed with a strict group law, or alternatively a category internal to the category of groups. As these objects are less well known, we will briefly summarise their main properties. We will use the terminology 'cat^1-group' introduced for these objects by Loday [17].

A cat^1-group is a group, G together with a subgroup N and two homomorphisms, $s, t : G \to N$ such that
(i) $s|_N = t|_N = id_N$
(ii) $[Ker s, Ker t] = 1$.

The group G is thought of as the collection of arrows of the (internal) category, whilst N is the group of objects, then s and t are the source and target homomorphisms. Axiom (i) then interprets as stating that identity 'arrows' start and finish at the same 'object', whilst (ii) is a subtle way of expressing the fact that composition of 'arrows' is a homomorphism. The category structure is, in fact, a groupoid. Taking the nerve of that groupoid gives a simplicial group $G_\bullet$ with $G_1 = G$ and $G_0 = N$, $d_0 = t$, and $d_1 = s$.

Examples

The concept of a cat^1-group is equivalent to that of a crossed module. Both aspects arise naturally, so we briefly look at these as well.

A *crossed module* consists of a pair of groups M, P and a homomorphism $\mu : M \to P$ together with an action of P on M (here written on the left). This data is to satisfy two axioms:

CM1) μ is P-equivariant, i.e.

$$\mu({}^p m) = p\mu(m)p^{-1}$$

with P acting on itself by conjugation;

CM2) (Peiffer identity)

$$^{\mu m}m' = mm'm^{-1} \quad \text{for all } m, m' \in M.$$

We note:

- Any normal subgroup $M \lhd P$ gives a crossed module with μ the inclusion; conversely, if (M, P, μ) is a crossed module then $\mu(M)$ is a normal subgroup of P.
- Any P-module M gives a crossed module in which $\mu(M) = 1$, the identity element of P; conversely any (M, P, μ) has $ker\mu$ a P-module.

Finally one of the most important examples is :

- if G is a group, take $P = Aut(G)$, $M = G$ and μ to be the morphism sending g to the inner automorphism corresponding to conjugation by g. (The importance of this example for the development of non-abelian cohomology is discussed by Breen, [2], [3] and [4].)

Of course, all three examples clearly exist in finite cases, as required for Yetter's construction to work.

For any simplicial group, $G_\bullet$, its Moore complex, $(NG_\bullet, \partial)$, is a chain complex of groups with

$$NG_n = \bigcap_{i \neq 0} Ker(d_i : G_n \to G_{n-1}),$$

$\partial_n : NG_n \to NG_{n-1}$ being given by the restriction of d_0 to NG_n, (cf. May, [18], or more briefly, Curtis, [10]).

The particularity of the nerve of a cat^1-group, as considered above, is that $NG_\bullet$ is there the trivial group in dimensions greater than 1 and the final boundary map,

$$\partial : NG_1 \to NG_0,$$

is a crossed module from which the cat^1-group can be rebuilt, up to isomorphism.

Proposition 1.2 *Let* $\mathbf{T}$ *be an ordered triangulation of a space,* X *and let* $G_\bullet$ *be a simplicial group with* $NG_n = 1$ *for* $n \geq 2$ *(so that* $G_\bullet$ *is isomorphic to the nerve of a cat*^{1}*-group, which will be denoted,* G*). Suppose that* λ *is a* G*-coloring of* T *in the sense of [21], p.116, then* λ *corresponds to a morphism of simplicial groupoids,*

$$\lambda' : G(T) \to G_\bullet,$$

where λ *is given on generators of* $G(T)$ *by*

$$\lambda'_0 < a_0, a_1 >= \lambda < a_0, a_1 >,$$

$$\lambda'_1 < a_0, a_1, a_2 >= \lambda < a_0, a_1, a_2 > s_0\lambda < a_1, a_2 >^{-1},$$

and

$$\lambda'_2 < a_0, a_1, a_2, a_3 >= \\ s_0\lambda < a_0, a_2, a_3 > \quad s_1 s_0 \lambda < a_2, a_3 >^{-1} s_1 s_0 \lambda < a_0, a_2 >^{-1} \\ s_1\lambda < a_0, a_1, a_2 > s_1 s_0 \lambda < a_2, a_3 > s_0 \lambda < a_1, a_2, a_3 >^{-1}$$

with higher dimensions determined by these.

Proof

Note first that as NG is of length at most 1, all higher dimensions of $G_\bullet$ are generated by degenerate elements, coming from dimensions 0 or 1 of $G_\bullet$. Any simplicial map

$$\phi : G(T) \to G_\bullet,$$

thus factors through the 2-coskeleton or 2-truncation of $G(T)$. This justifies the final statement of the proposition. The mystifying nature of the definition of λ'_2 corresponds to Yetter's condition: "the faces of every ordered 3-simplex commute as a diagram in G".

The remainder of the proof consists in checking that the face relations hold between these. This is a routine calculation and so will be omitted. □

Given λ', it is clear that one can retrieve λ and hence that one can consider $\Lambda_G(T)$ to be the set of simplicial groupoid maps from $G(T)$ to $G_\bullet$.

Remark. It is clear that there are technical advantages in considering a more general situation, namely replacing the 2-type represented by $G_\bullet$ by an arbitrary (finite) simplicial group or groupoid. In a sequel to this paper, we will give a detailed discussion on how this can be done.

2 Interpretation of G-colorings

In this section, we will examine the interpretation of G-colorings of a fixed triangulation $\mathbf{T}$, where G is a cat^1-group or equivalently a simplicial group with Moore complex of length at most 1.

2.1 Crossed module maps

Given a simplicial groupoid, H, we have a set of objects $O(H)$ and in each dimension n, there is a set of identities $\{1^n_x : x \in O(H)\}$. Given any morphism, $\phi : A \to B$, of groupoids (over a fixed set, O, of objects, so that $\phi(1_x) = 1_x$ for each $x \in O$), it is routine to define $ker\phi$ to be the set of elements of A sent to identities by ϕ. This defines a normal subgroupoid of A, which takes the form of a collection of normal subgroups of the vertex groups of the groupoid A,

$$ker\phi = \coprod_{x \in O} ker\{\phi_x : A(x,x) \to B(x,x)\}.$$

(This is a disjoint union <u>not</u> a coproduct of groups. It is in fact a coproduct, but in the category of groupoids.) Given this it is easy to generalise most of the theory of Moore complexes from simplicial groups to simplicial groupoids.

Formally, if H is a simplicial groupoid,

$$NH_n = \bigcap_{i=1}^{n} ker\{d_i : H_n \to H_{n-1}\}.$$

This Moore complex is almost just a disjoint union of the Moore complexes of the various $H(x,x)$, the vertex simplicial groups of H, however in dimension 0, it is the groupoid H_0 and so does in general have a collection of 0-arrows joining distinct objects. This, of course, implies that within any connected component of H, all $NH(x,x)$ are isomorphic.

Given any G-coloring, λ, the corresponding morphism,

$$\lambda : G(T) \to G,$$

will induce a morphism between the Moore complexes,

$$N(\lambda) : NG(T) \to NG.$$

Remark.
Technically these Moore complexes carry the structure of hypercrossed complexes, that is chain complexes of group(oid)s with pairings and actions specified (see Carrasco and Cegarra, [9]). As will be clear shortly, we will not need the higher order structure in our situation, and will retain only the chain complex structure with the actions of the zero-level groupoid.

As the Moore complex, NG, of G is of length 1, $\lambda(NG(T))_n = 1$ for all $n > 1$. This implies that $N(\lambda)$ factors through the quotient of $NG(T)$ given by 'killing off' all $NG(T)_n$ for $n > 1$. This corresponds to a crossed module (of groupoids)

$$\frac{NG(T)_1}{\partial NG(T)_2} \to NG(T)_0,$$

which will be denoted $\Pi_2(T)$. As explained here, this construction is not very direct, but it emphasises the fact that

$$SGpds(G(T), G) \cong CMod(\Pi_2(T), NG).$$

This crossed module $\Pi_2(T)$ has several other descriptions. The most classical one is obtained by noting that $NG(T)_0 \cong \Pi T_1 T_0$, the fundamental groupoid of the 1-skeleton of T, based at the vertices of T, thus $NG(T)_0$ is a free groupoid. The groupoid $\pi_2(T)_1$ in dimension 1 of the crossed module is the disjoint union of the relative homotopy groups, $\pi_2(T, T_1, v)$, where as before T_n denotes the n-skeleton of T, and v is a vertex of the triangulation.

Remark.
Given that n-categorical machinery is currently being evaluated for its potential use in quantum field theory, it should be noted that this type of crossed module corresponds to a certain class of double category (with connections) and also to small 2-categories constructed geometrically from the filtered space, X. We refer the interested reader to the various survey articles of R. Brown, (see, for instance, [5] combined with that with Huebschmann, [8]).

We summarise the above in the following proposition:

Proposition 2.1 *There is a natural bijection between the set, $\Lambda_G(\mathbf{T})$, of colorings of the triangulation $\mathbf{T}$ in the (finite) cat^1-group, G, and the set of crossed module morphisms from $\Pi_2(T)$ to the crossed module associated to G.*

Before passing to the next interpretation, it is worth noting that $CMod(\Pi_2(T), NG)$ is the set of objects of a groupoid that serves as the base of a crossed module. In fact, any crossed module can be considered as a crossed complex. (Roughly a crossed complex is a non-negative chain complex of group(oid)s, abelian above dimension 1, with bottom morphism a crossed module, see Brown and Higgins, [6].) The category of crossed complexes is monoidal closed with 'internal hom' $CRS(\_,\_)$. In this case $CRS(\Pi_2(T), NG)$ is a crossed complex with $CMod(\Pi_2(T), NG)$ as its base. The category of crossed complexes is equivalent to various variants of the category of ω-groupoids (again see references to the work of Brown and Higgins, e.g. [7]). The crossed complex in question here is effectively just a groupoid as it is trivial in higher dimensions, however this suggests that a theory, related to that of Yetter, but with $Z_G(\mathbf{T})$ replaced by the groupoid algebra, $\mathbb{C}[CRS(\Pi_2(T), NG)]$ might be worth investigating. It should also be noted that the arrows of $CRS(\Pi_2(T), NG)$ have a geometric interpretation, but this will not be considered here.

2.2 Simplicial fibre bundles with groupoid fibres.

As in the case of G, a finite group, considered earlier, any G-coloring

$$\lambda : G(\mathbf{T}) \to G$$

corresponds by adjointness to a simplicial morphism,

$$\overline{\lambda} : T \to \overline{W}G.$$

Again, on $\overline{W}G$, there is a principal G-bundle,

$$WG \to \overline{W}G,$$

whose fibre is the underlying simplicial set of G, which will be denoted $U(G)$. The morphism $\overline{\lambda}$ induces by pullback a principal fibre bundle on T, again with $U(G)$ as fibre.

The simplicial set $U(G)$ still retains algebraic structure. The simplicial group G was introduced as being the nerve in the groupoid direction of the cat^1-group, also denoted G. In $U(G)$, we have forgotten the group structure, but the implicit structure of a groupoid is still present.

For any groupoid, H, the nerve has a special characteristic property. Let $n \geq 0$ and $0 \leq i \leq n$, the $\bigwedge^i[n]$ will denote the standard (n, i)-horn, that is, the subcomplex of the standard n-simplex $\Delta[n]$ determined by all the $(n-1)$-faces except the i^{th} one. If K is any simplicial set, a (n, i)-horn in K is a simplicial map, $x : \bigwedge^i[n] \to K$, and corresponds to a collection $(x_k)_{k\neq i}$ of $(n-1)$-simplices in K which satisfy

$$d_j x_k = d_{k-1} x_j \qquad j < k,$$

but note no x_i is present in the collection. The set of (n, i)-horns in K is thus the set $SS(\bigwedge^i[n], K)$, of simplicial maps from $\bigwedge^i[n]$ to K.

Lemma 2.2 *(cf. Illusie, [15], or Glenn, [14] p.42)*
A simplicial set, K is isomorphic to $NerH$ for some groupoid H if and only if for all $n \geq 2$ and all $0 \leq i \leq n$, the natural map

$$K_n \to SS(\overset{i}{\bigwedge}[n], K)$$

is an isomorphism □

If G is a nerve of a cat^1-group, then not only is it a simplicial group, but its underlying simplicial set also satisfies the condition of the above lemma, i.e. the algebraic (groupoid) structure is encoded in the simplicial structure. Thus the induced bundle $E(\overline{\lambda}) = WG^{\overline{\lambda}}$ has fibres which are (finite) groupoids.

To understand more fully these fibre bundles, we will briefly turn aside from the main discussion to consider their special properties.

Covering spaces are fibrations with discrete fibres and they satisfy a unique path lifting property. This can be paraphrased as saying that if $p : E \to B$ is a covering projection and $\omega : [0,1] \to B$ is a path, then if we state where the lifted path is to start (i.e. we choose $e_0 \in E$, $p(e_0) = \omega(0)$), then there is one and only one lift of ω starting at e_0. This is, of course, 'classical' but provides a comparison for the properties of $WG^{\overline{\lambda}}$.

Study of the induced bundle $WG^{\overline{\lambda}}$ for $\lambda : G(T) \to G$.

We will assume that G is the simplicial group associated as above to a cat^1-group of the same name, and corresponding to a crossed module $\partial : C \to P$, so $G_0 = P$, $G_1 = C \rtimes P$, $G_2 = C \rtimes (C \rtimes P)$, etc. It is equivalent to use the induced mapping $\overline{\lambda} : T \to \overline{W}G$ as base change to pull back WG to $E(\overline{\lambda}) = WG^{\overline{\lambda}}$, or to use λ as a twisting function for the construction of E as a twisted cartesian product (TCP), see May, [18] or Curtis, [10], but beware, we have used a different convention for the d_0 in $G(T)$, which changes the order in one or two places.

In the TCP approach, $E = G \times_\lambda T$ where each E_n is just the product $G_n \times T_n$, all the d_i for $i > 0$ and s_i for $i \geq 0$, are the usual ones for a cartesian product, but the 0-face map is twisted by λ :

$$d_0(a, x) = (d_0 a.\lambda x, d_0 x).$$

Remark

Of the various equivalent forms of λ that we can use, here it is perhaps easiest to think of λ as being a family

$$\lambda_n : T_n \to G_{n-1}$$

satisfying

$$d_i\lambda(x) = \lambda(d_{i+1}x) \qquad i > 0$$

$$s_i\lambda(x) = \lambda(s_{i+1}x) \qquad i \geq 0$$

$$\lambda(s_0 x) = e_q,$$

the identity element of G_q if $x \in T_q$, and

$$d_0\lambda(x) = \lambda(d_1 x)\lambda(d_0 x)^{-1}.$$

It may help to compare these with the descriptions of the face and degeneracy operators in $G(T)$ and $\overline{W}G$.

As an example that is relevant later on, consider the case when G is a cat^1-group associated to $\partial : C \to P$. This gives, for instance,

$$E_2 \cong (C \times C \times P) \times T_2$$

as a set, i.e. ignoring group structures, with, for example,

$$d_0((c_2, c_1, p), x) = ((c_2, \partial c_1.p).(\lambda_1 x), d_0 x).$$

If we write $\lambda_1(x) = (\lambda_1^C(x), \lambda_1^P(x)) \in C \rtimes P$, then the conditions on λ give $\lambda_1^P(x) = \lambda_0(d_2x)$, whilst $\lambda_1^C(x)$ satisfies

$$\partial\lambda_1^C(x) = \lambda_0(d_1x)\lambda_0(d_0x)^{-1}\lambda_0(d_2x)^{-1}.$$

Thus

$$\begin{aligned} d_0((c_2, c_1, p)x) &= ((c_2.{}^{\partial c_1.p}\lambda_1^c x, \partial c_1.p.\lambda_0(d_2x)), d_0x) \\ &= ((c_2.c_1.{}^p\lambda_1 x.c_1^{-1}, \partial c_1.p.\lambda_0(d_2x)), d_0x). \end{aligned}$$

The complexity of the result is partially due to the multiplication in the semidirect product, but, as it will be seen later, direct calculations are still relatively easy to do. Finally, of course, the projection, p, is the obvious projection onto T, $p(a, x) = x$.

Proposition 2.3 *Suppose that $\partial : C \to P$ is a finite crossed module, and G is the corresponding cat^1-group. Let $p : E \to T$ be a principal G-bundle, induced by λ say, then*
a) given any edge, $x \in T_1$ and any $y \in E_0$ with $p(y) = d_1x$, there are $\sharp(C)$ lifts of x starting at y;
b) given any $x \in T_2$ and $y : \bigwedge^1[2] \to E$ so that

$$p(y) = (d_2x, -, d_0x)$$

then there is a unique lift of x to some $z \in E_2$ with

$$y = (d_2z, -, d_0z).$$

Remark.

The proposition can be interpreted as saying that edges lift within $\sharp(C)$ copies, but 'homotopies' lift uniquely. (We say 'edges' not paths as T will in the applications not be a Kan complex in general.) This suggests that such principal G-bundles may be a reasonable first 'laxification' of the notion of covering space.

Proof of Proposition 2.3

Both parts are proved by direct calculation. Part (a) being easy, we will concentrate on (b). Write $y = (y_2, -, y_0)$, $y_0, y_2 \in C \times P \times T_1$, then $y_2 = ((c, p), d_2x)$, $y_0 = ((c', p'), d_0x)$ and the compatibility conditions shows that $p' = \partial c.p.\lambda_0(d_2x)$.

Now suppose z fills y, so $y_0 = d_0z$, $y_2 = d_2z$, and covers x, then from this one finds that if

$$z = ((c_2, c_1, p_0), x),$$

then $(c_1, p_0) = (c, p)$ and $c' = c_2.c_1.{}^p\lambda_1(x)c^{-1}$. Thus given y_2 and y_0, there is a unique such z namely

$$z = ((c'c^p\lambda_1(x)^{-1}c^{-1}, c, p), x).$$

□

We might tentatively call such a bundle a '1-lax covering space'. Although '1-stack' may be a better term, (see Breen [4]), the precise relationship between stacks of groupoids and these bundles is not yet clear. Some initial results are given below however.

Within the framework of G-colorings and simplicial fibre bundles, it is worth noting that if $\lambda, \lambda' : T \to \overline{W}G$ are two G-colorings, then if $\lambda \simeq \lambda'$, the corresponding bundles are G-equivalent.

2.3 Non-abelian descent data and cocycles.

The link between G-fibre bundles and non-abelian cohomology with coefficients in G is well known and classical. Recent advances in non-abelian cohomology suggest that more general forms of cohomology correspond to more 'lax'-versions of G-bundles such as G-gerbes, stacks, etc. (cf. Breen, [4]) and hence the interpretation here may lead to other TQFTs on non-abelian cohomology sets having geometric interpretations.

The usual approach to the connection with cohomology is via an open cover, $\mathcal{U}$, of the space, X, and the sections of the fibre bundle over the sets of the form $U_{\alpha_0} \cap \ldots U_{\alpha_n}$, which are non-empty. The nerve of the open cover then gives a simplicial set representing the combinatorial information about these intersections. As was mentioned earlier, the triangulation, $\mathbf{T}$, or more exactly the simplicial complex part of the triangulation, is isomorphic to the nerve of the open cover determined by the open stars of the vertices in $\mathbf{T}$, thus linking triangulations with open covers.

We will be looking at cocycles with values in G. If the reader compares with Breen, [2], then G is here being thought of as a constant sheaf of cat^1-groups. Duskin, [11] adopts a very similar viewpoint to Breen. He would consider G as a 2-group, that is a 2-groupoid with a single object, where a 2-groupoid is a 2-category in which every 1-cell is a formal 'equivalence of categories' and every 'natural transformation', i.e. 2-cell, is an isomorphism.

Both Breen and Duskin then define cocycles to be what, when translated to the context here, just amounts to a G-coloring. To be more explicit, let $\mathbf{C} = \acute{E}t(X)$ be the category of étale spaces over X, so that an open cover $\mathcal{U}$ can be thought of as a 'covering map' $X(\mathcal{U}) = \coprod U_\alpha \to X$ and hence as an object in $\mathbf{C}$. Repeated pullbacks yield a simplicial object, $X_\bullet \to X$ in $\mathbf{C}$, augmented over the terminal object $Id_X : X \longrightarrow X$. This simplicial object is 'almost' $Ner(\mathcal{U})$, but is not just a simplicial set, as each simplex $< U_0, \ldots, U_n >$ of the nerve is here thought of as a space étale over X, namely $U_0 \cap \ldots \cap U_n \to X$.

For descent data, in Duskin's terminology, one takes a functor $F : \mathbf{C}^{op} \to \mathbf{2-CAT}$, which in our case is constant with value G considered as a 2-group (see above). Duskin then defines (p.257), a 2-category $\underline{DES}^{(2)}(X_\bullet/X; F)$ of (2-)descent data on $X_\bullet$ with coefficients in F as follows (with comments on the interpretation in our much simpler situation) :

Its objects (i.e. 0-cells) are triplets (α, f, x) consisting of

a) an object (0-cell) x of the 2-category FX_0, (there is not much choice for us as $FX_0 = G$ has one exactly object!);

b) a 1-cell f in FX_1 of the form $f : Fd_1(x) \to Fd_0(x)$, (for us, $d_i : X_1 \to X_0$, $i = 0, 1$, correspond to the two maps given by the inclusions $U_0 \cap U_1 \to U_1$ and $U_0 \cap U_1 \to U_0$, (note the reversal of order), so if the covering corresponds to the triangulation $\mathbf{T}$, and $U_i =$ the open star of vertex v_i, then $U_0 \cap U_1$ 'is' an edge from v_0 to v_1 in the nerve and f corresponds to $\lambda < v_0, v_1 >$ as a 1-cell in the 2-group, G);

c) a 2-cell α in FX_2 of the form

$$\alpha : Fd_1(f) \Rightarrow Fd_0(f)Fd_2(f),$$

subject to a tetrahedral (2-cocycle) condition, (α is an isomorphism; if we invert it, we find a possible $\lambda < v_0, v_1, v_2 >$ in Yetter's notation, and his 'ordered 3-simplex' condition (p. 116) is then exactly Duskin's tetrahedral condition (p. 258)).

Remark.

For the conditions on α, we have already noted the equivalence of this with simplicial maps from $G(T)$ to G or from T to $\overline{W}G$. A glance at Duskin (p.269) shows that this was already implied by his work.

We have thus shown that

Proposition 2.4 *Suppose that G is a (finite) cat^1-group and F is the constant sheaf with value G on $\mathbf{C} = \acute{E}t(X)$, the category of spaces étale over X, then G-colorings of T correspond bijectively to normalised 2-descent data on the open star cover of T with coefficients in F.*

A remark on terminology is necessary. The use of terms such as 'n-cocycle' within non-abelian cohomology and descent theory does not yet seem to have stabilised in meaning for various good reasons. For example, the seminal work of Breen, [4], talks of 2-cocycles with coefficients in a sheaf of groups, G or in a gr-stack on X, but the 2-cocycles in Duskin's data, [11], would in this case have coefficients in the automorphism cat^1-group of G. It is for this reason that we have used 2-descent data as the key idea rather than 2-cocyles. The relationship is, of course, very strong.

In our situation, the 2-cocycles seem to be almost 'strict' as they correspond to an actual (simplicial) fibre bundle. The possibility of 'laxifying' the coefficient structure further might be worth investigating from the point of view of generating interesting new TQFTs.

We postpone discusssion of the 1-cells and 2-cells in $\underline{DES}^{(2)}(X_\bullet/X;F)$ until we have studied the dependence of Yetter's colorings on subdivision.

3 Subdivisions

The above interpretations are all dependent on a given triangulation, $\mathbf{T}$ or open cover of X. Yetter's construction in both of the papers, [20] and [21], studies the way in which the colorings of a subdivision of $\mathbf{T}$ relate to those of $\mathbf{T}$ itself, then passing to a (co)limit over all triangulations allows the construction to be made independent of the choice of triangulation. We will need to analyse Yetter's construction in some detail so as to examine the end result for our various interpretations.

3.1 Yetter's construction

Recall that $\Lambda_G(T)$ is the set of all G-colorings of $\mathbf{T}$. We first describe the case when G is a finite group before adapting it to the cat^1-group case. Let $Z_G(X,\mathbf{T})$ denote the vector space (over $\mathbb{C}$) with $\Lambda_G(\mathbf{T})$ as basis. Suppose that $\mathbf{T}'$ is a subdivision of $\mathbf{T}$ obtained by (iterated) subdivision of edges. The map $\mathrm{res}_{\mathbf{T}',\mathbf{T}}$ from $\Lambda_G(\mathbf{T}')$ to $\Lambda_G(\mathbf{T})$ is given by

$$\mathrm{res}_{\mathbf{T}',\mathbf{T}}(\eta)(\mathrm{e}) = \prod \eta(\mathrm{e_i}),$$

where the edge, e, of T has been subdivided to give $e_1 \ldots e_n$. This induces a map, also denoted $\mathrm{res}_{\mathbf{T}',\mathbf{T}}$ from $Z_G(X,\mathbf{T}')$ to $Z_G(X,\mathbf{T})$. To ensure compatibility with the maps coming from cobordisms (which will be considered later), Yetter uses a weighted form of 'res' defined by

$$res_{\mathbf{T}',\mathbf{T}} = \sharp(G)^{-\frac{1}{2}(\sharp(T'_{(0)})-\sharp(T_{(0)}))}\mathrm{res}_{\mathbf{T}',\mathbf{T}}$$

whenever $\mathbf{T}'$ is obtained from $\mathbf{T}$ by iterated subdivision. The $Z_G(X,\mathbf{T})$s and $res_{\mathbf{T}',\mathbf{T}}$s form a diagram of vector spaces over the partially ordered set of triangulations 'ordered by inclusion'. The exact meaning of this is not made explicit, but it is clear that considering this poset as a category, we need $T' \to T$ so that $res_{\mathbf{T}',\mathbf{T}} : Z_G(X, \mathbf{T}') \to Z_G(X, \mathbf{T})$ corresponds to a functor defined on the 'poset'. (This direction is somewhat at variance with the tradition of 'passage to the limit', for instance in Čech cohomology, in which the limit is taken in the direction of refinement, but it is related as we will see later.) Yetter defines $Z_G(X)$ to be the colimit of this diagram. This removes the dependence on $\mathbf{T}$.

For any non-empty manifold, X, the $res_{\mathbf{T}',\mathbf{T}}$ are all epimorphisms, so as the poset of triangulations is cofiltered (any two triangulations admit a common subdivision), the canonical maps

$$g^X_{\mathbf{T}} : Z_G(X, \mathbf{T}) \to Z_G(X)$$

are all epimorphisms, and hence $Z_G(X)$ is finite dimensional. As Yetter comments (p.6 of [20]) *'To understand the colimit, $Z_G(X)$, it thus suffices to understand the kernel of a single $g^X_{\mathbf{T}}$.'* If $\xi \in Z_G(X, \mathbf{T})$ is in the kernel of $g^X_{\mathbf{T}}$, then by the construction of colimits of vector spaces, there must be a triangulation $\mathbf{T}'$ and a joint subdivision $\mathbf{T}''$ of both $\mathbf{T}$ and $\mathbf{T}'$, together with some $\xi'' \in Z_G(X, T'')$ such that

$$res_{\mathbf{T}'',\mathbf{T}}(\xi'') = \xi,$$

and

$$res_{\mathbf{T}'',\mathbf{T}'}(\xi'') = 0.$$

Of course, if $c = \sharp(G)^{\frac{1}{2}(\sharp(T'_{(0)}) - \sharp(T_{(0)}))}$, then $\mathrm{res}_{\mathbf{T}'',\mathbf{T}}(\mathrm{c}\xi'') = \xi$ and $\mathrm{res}_{\mathbf{T}'',\mathbf{T}'}(\mathrm{c}\xi'')$ is still zero, thus if we denote by $Z'_G(X)$, the colimit of the diagram with $Z_G(X, \mathbf{T})$s still, but with $\mathrm{res}_{\mathbf{T}',\mathbf{T}}$ replacing $res_{\mathbf{T}',\mathbf{T}}$, then we have

Lemma 3.1 *There is a natural isomorphism*

$$Z_G(X) \cong Z'_G(X).$$

□

One still needs the *res*-maps for ease of handling the compatibility with the maps coming from the cobordisms, but the above lemma greatly simplifies the task of describing $Z_G(X)$, especially as $Z'_G(X)$ is isomorphic to the vector space with basis $\Lambda_G(X) = colim\{\Lambda_G(\mathbf{T}), \mathrm{res}_{\mathbf{T}',\mathbf{T}}\}$, as formation of a vector space is a left adjoint and so commutes with colimits.

Remarks.

(i) The essence of the idea behind 3.1 is given by Yetter's proof of his lemma 2.9, p.9 of [20].

(ii) To adapt the above to the case when G is a finite cat^1-group, associated to a crossed module $\mathcal{M} = (\partial : C \to P)$, one uses the formulae on p.117 of [21] to define a res-map and then uses the weighting factor involving $g_0 = \sharp(P)$ and $g_1 = \sharp(C)$. The exact formula is

$$res_{T',T} = g_0^{-\frac{1}{2}(\sharp(T'_{(0)}) - \sharp(T_{(0)}))} g_1^{-\frac{1}{2}(\sharp(T'_{(1)}) - \sharp(T'_{(0)}) - \sharp(T_{(1)}) + \sharp(T_{(0)}))} \mathrm{res}_{\mathrm{T}',\mathrm{T}}$$

The geometric significance of these slightly strange numbers should become clear later on. Of course the argument leading to Lemma 3.1 and the simplified description of $Z_G(X)$ is still valid.

3.2 Subdivisions and the loop-groupoid construction.

As Yetter points out, his results are for a $(2+1)$-TQFT, as he needs results on triangulations and subdivisions which, in general, need more careful handling in higher dimensions. The necessity for this restriction and also for his restriction to edge-stellar subdivisions, is not clear since, as we have seen, some of the interpretations of colorings fit more neatly within a framework of open coverings and their nerves. We will see that the definition of his res-map is possible in arbitrary dimensions and for arbitrary subdivisions. (As a source for simplicial subdivisions, we use Spanier, [19], Chap. 3, Sec. 3.)

Given a triangulation of X, we may subdivide $\mathbf{T}$ by picking some simplex, $s = < v_0, \ldots, v_n >$ in T and some point $\alpha \in |s|$. (We will assume for convenience that the simplicial complex is ordered.) As $\alpha \in |s|$, we must have $\alpha \in < s' > = \{v \in \{v_0, \ldots, v_n\} | \alpha(v) \neq 0\}$, the carrier of α. For simplicity, we will assume that n= dim X and $s' = s$, i.e. that the 'new vertex' is in the open simplex $< s >$.

One way to view a subdivision is as a cone on a simplex (cf. Spanier, [19], p.123). Take ∂s to be the boundary of s and $\alpha \in < s >$, the join $\partial s * \alpha$ gives a subdivision of s that can be extended to a subdivision of T by repeating the construction inductively up the skeleta of T. (This latter point will only be needed if $n < dim X$.) We take the cone point, denoted v_α, as a new vertex. We pick an ordering on $|T_0| \cup \{v_\alpha\}$ extending that on $|T_0|$, to get an ordered triangulation $\mathbf{T}'$ of X. In general, of course, this process needs repeating to get finer and finer subdivisions of $\mathbf{T}$, but we use the above to study the effect of this 'inductive step' on $G(T)$.

If one takes α and forms the cone $s * \alpha$, then one replaces s by an (n+1)-simplex, σ, with s as one of its faces, which face will depend on the relative position of the new vertex v_α in the ordering of the vertices of $\mathbf{T}'$. Explicitly, if the new ordering is $v_0 < v_1 < \ldots < v_i < v_\alpha < v_{i+1} < \ldots < v_n$, then $s = < v_0, \ldots, v_n > = d_{i+1}\sigma$. Using this one obtains within $G(T')$, a (n,i)-horn which can be filled using the usual filling algorithm for horns in simplicial group(oid)s (cf. May, [18]; beware the error in Curtis, [10], at this point.) The ith-face of this filler algebraically models the simplex in $G(T)_{n-1}$ corresponding to s. (If $v_\alpha < v_0$, some care needs to be taken since the d_0 of $G(T')$ involves both d_1 and d_0 of T'.) This means that there is strong deformation retraction data :

$$\mathbf{r}_T^{T'} : G(T) \to G(T'),$$

$$\mathbf{s}_{T'}^{T} : G(T') \to G(T),$$

with $\mathbf{s}_{T'}^{T}\mathbf{r}_T^{T'} = Id_{G(T)}$, but $\mathbf{r}_T^{T'}\mathbf{s}_{T'}^{T} \simeq Id_{G(T')}$. Here $\mathbf{r}$ gives the pasted composite of the faces of the simplices of T' that make up the subdivision of s. This situation can be iterated so that it still holds if $\mathbf{T}'$ is any subdivision of $\mathbf{T}$. The map $\mathbf{s}_{T'}^{T}$ can be chosen to be induced from a simplicial approximation, $\overline{\mathbf{s}}_{T'}^{T}$, to the identity on X. There are many different choices of $\overline{\mathbf{s}}_{T'}^{T}$, but they are all contiguous and thus homotopic to each other by explicit homotopies.

Lemma 3.2 *If $\mathbf{T}'$ is obtained from $\mathbf{T}$ by iterated edge-stellar subdivision, then for any finite group or cat^1-group, G,*

$$\mathrm{res}_{\mathrm{T}',\mathrm{T}} : \Lambda_{\mathrm{G}}(\mathrm{T}') \to \Lambda_{\mathrm{G}}(\mathrm{T})$$

is given by $\mathrm{res}_{\mathrm{T}',\mathrm{T}}(\lambda) = \lambda \cdot \mathbf{r}_{\mathrm{T}'}^{\mathrm{T}}$ modulo the identification of $\Lambda_G(T)$ as set of simplicial groupoid maps from $G(T)$ to G. □

The proof merely examines the single subdivision of an edge for which direct formulae are immediately verifiable.

Notice that if one wishes to verify the earlier statement that $\mathrm{res}_{\mathrm{T}',\mathrm{T}}$ is onto, it is now simple. If $\lambda \in \Lambda_G)(T)$, set $\lambda' = \lambda \cdot \mathbf{s}^T_{T'}$ and then $\mathrm{res}_{\mathrm{T}',\mathrm{T}}(\lambda') = \lambda$. Of course, there is usually more than one G-coloring of T' mapping to λ.

Because of this lemma, we shall not assume that subdivisions are necessarily obtained by iterated edge-stellar operations as this latter method is occasionally restrictive and, in general, we will use the formula of the lemma to define $\mathrm{res}_{\mathrm{T}',\mathrm{T}}$.

Returning to the strong deformation retraction (sdr) data, analysis of the filling algorithm used shows what should be geometrically obvious. The homotopy $rs \simeq Id_{G(T')}$ is filtered in the following sense:

Suppose $n \geq 1$ and that T' is obtained from T by taking as above $s \in T_n$, and forming $\partial s * \alpha$ for some $\alpha \in< s >$ (i.e. a single simplex subdivision extended over T.) This gives an (n, i)-horn in $G(T')$ but this horn factors,

$$\bigwedge{}^i[n] \to sk_{n-1}G(T') \subset G(T'),$$

where $sk_{n-1}G(T')$ is the subsimplicial groupoid generated by all $G(T')_k$, for $k \leq n-1$. In dimensions bigger than n, $sk_{n-1}G(T')$ consists of composites of degenerate elements $s_\alpha(x)$ with the dimension of x less than or equal to $n-1$. (Note a composite such as $s_0x \cdot s_1y$ will not be degenerate itself although it is a composite of degenerate elements.) The observation is that the data for filling the above horn is all in $sk_{n-1}G(T')$ and contains no 'new' information whose geometric origin has higher dimension, i.e. the natural filler gives

$$\Delta[n] \to sk_{n-1}G(T')$$

and so as these fillers are the basis of the homotopy $\mathbf{rs} \simeq Id_{G(T')}$, this homotopy is relative to the skeletal filtration of $G(T')$, that is if $n \geq 1$, the images of all the $(n-1)$-simplices use only $sk_{n-1}G(T')$. We summarise this for future refence:

Lemma 3.3 *The homotopy from $\mathbf{r}^{T'}_T \mathbf{s}^T_{T'}$ to the identity is relative to the filtration of $G(T')$ by its skeleta.* □

As our understanding of homotopies of simplicial groupoids is relatively poor, we will also need to view the above in the adjoint setting. The simplex $s \in T_n$, $(n \geq 1)$ will be subdivided giving an $(n+1, i+1)$-horn that can be filled within $\overline{W}sk_{n-1}G(T') \subset \overline{W}G(T')$. This gives a representation of $\mathbf{r}$ as

$$\overline{\mathbf{r}} : T \to \overline{W}G(T').$$

(The relationship with the groupoid version is given by the adjointness relation. If

$$\eta_T : T \to \overline{W}G(T)$$

is the unit of the adjunction, then $\overline{\mathbf{r}}$ is the composite

$$T \xrightarrow{\eta_T} \overline{W}G(T) \xrightarrow{\overline{W}(\mathbf{r})} \overline{W}G(T').$$

Now letting $\overline{\mathbf{s}}$ be a chosen simplicial approximation to the identity,

$$\overline{\mathbf{s}} : T' \to T$$

so that $\mathbf{s} = G(\overline{\mathbf{s}})$, the fact that $\mathbf{sr}$ is the identity on $G(T)$ means that $\overline{W}(\mathbf{s})\overline{\mathbf{r}} = \eta_T$, whilst

$$\begin{aligned} \overline{\mathbf{r}\mathbf{s}} &= \overline{W}(\mathbf{r})\eta_T\overline{\mathbf{s}} \\ &= \overline{W}(\mathbf{r})\overline{W}(\mathbf{s})\eta_{T'} \\ &= \overline{W}(\mathbf{rs})\eta_{T'} \\ &\simeq \eta_{T'} \qquad \text{by a filtered homotopy} \end{aligned}$$

The meaning of 'filtered homotopy' here is that if T' is filtered by skeleta, $sk_n(T')$, and $\overline{W}G(T')$ is filtered by the $\overline{W}sk_nG(T')$, then the homotopy $h : T \times I \to \overline{W}G(T)$ does not raise filtration, thus for instance, $h : sk_nT \times I \to \overline{W}sk_nG(T)$. One final and important feature of the homotopy is that if T' is obtained from T by subdividing edges, then on the old vertices of T, the homotopy can be chosen to be constant.

Although the converse of the above is true, the technicalities of the statement are such as to make it clearer when discussing G-colorings, we therefore will put off until later the discussion of that result.

3.3 Subdivision and G-colorings

We filter $\overline{W}G$ by the $\overline{W}sk_nG$.

Proposition 3.4 *Suppose $\lambda : T \to \overline{W}G$, $\lambda' : T' \to \overline{W}G$ are two G-colorings such that $g_T^X(\lambda) = g_{T'}^X(\lambda')$, then there is a joint subdivision T'' of T and T', and simplicial approximations to the identity, $\mathbf{s} : T'' \to T$, $\mathbf{s}' : T'' \to T'$ such that $\lambda \circ \overline{\mathbf{s}}$ and $\lambda' \circ \overline{\mathbf{s}'}$ are filtered homotopic. If $T_0 \cap T_0'$ is non-empty the homotopy can be chosen to be constant on the vertices in this intersection.*

Proof.

We know that $g_T^X(\lambda) = g_{T'}^X(\lambda')$ if and only if there is some T'' and λ'' such that $res_{T'',T}(\lambda'') = \lambda$ and $res_{T'',T'}(\lambda'') = \lambda'$. Lemma 3.2 implies

$$\lambda \circ \overline{\mathbf{s}} = \lambda'' \circ \overline{\mathbf{r}} \circ \overline{\mathbf{s}},$$

but by 3.3 and the discussion after it $\overline{\mathbf{r}} \circ \overline{\mathbf{s}} \simeq Id$ by a filtered homotopy fixing old vertices. Repeating for λ' using $res_{T'',T'}(\lambda'') = \lambda'' \circ \overline{\mathbf{r}'}$, and $\overline{\mathbf{s}'}$ gives the result. □

Corollary 3.5 *Let λ, λ' be G-colorings as above, and $E(\lambda)$, $E(\lambda')$, the corresponding principal G-bundles. If $g_T^X(\lambda) = g_{T'}^X(\lambda')$ then there is a joint subdivision $\mathbf{T}''$ of $\mathbf{T}$ and $\mathbf{T}'$, and simplicial approximations to the identity*

$$s : T'' \to T$$

$$s' : T'' \to T'$$

such that $s^*(E(\lambda)) \cong s'^*(E(\lambda'))$. □

In other words, the fibre bundles corresponding to λ and λ' are, in some sense, locally isomorphic.

Identifying $\Lambda_G(T)$ with the set of maps from T to $\overline{W}G$, we obtain a surjection

$$\Lambda_G(T) \to [T, \overline{W}G].$$

Writing $\Lambda_G(X)$ for $colim(\Lambda_G(T), \mathrm{res}_{\mathrm{T}',\mathrm{T}})$, so that

$$Z_G(X) \cong \mathbb{C}(\Lambda_G(X))$$

by our earlier comments, the above proposition implies that:

Corollary 3.6 *There is a surjection*

$$\Lambda_G(X) \to colim([T, \overline{W}G], s^*).$$

$\square$.

Remark.

There is a need to be a bit cautious about the colimits here. By using the restriction map, Yetter builds $Z_G(X)$ so that the colimit is over triangulations and *inclusions*. This means that $Z_G(X)$ records information on the large scale triangulations, i.e. using few simplices. Two colorings are equivalent if they come from multiplying labels within a common subdivision. The colimit on the right is taken over the opposite category : triangulations with refinement or, more usefully, open covers with refinement, since as we have seen earlier $T \cong Ner(\mathcal{U})$, where $\mathcal{U}$ is the star open cover given by T. Here one is much nearer the classical construction of Čech cohomology. In fact if G is the simplicial group associated with the crossed module $A[1]$, that is $A \to 1$, where A is a finite abelian group, then

$$H^2(X, A) \cong Colim[T, \overline{W}A[1]]$$

by results of Breen, [2], p.438. Yetter in [21] proved that in this case, $\Lambda_G(X)$ is exactly $H^2(X, A)$, up to isomorphism. His motivation was to produce evidence that the TQFT that he had constructed was non-trivial and, of course, the above corollary acts as further evidence, as it extends the calculation to a larger class of cases.

This link with cohomology is strong, as in general, if G is the simplicial group associated with a finite crossed module, $\mathcal{M}$, then the colimit of the $[T, \overline{W}G]$ is what Breen, [2], defines as $H^1(\mathcal{M})$, but here, where its dependence on X needs recording, we will write as $H^1(X, \mathcal{M})$. The results on pages 437 and 438 of [2] give exact sequences relating $H^1(X, \mathcal{M})$ to the better known $H^2(X, \pi_1\mathcal{M})$ and $H^1(X, \pi_0\mathcal{M})$, where if $\mathcal{M} = (C \longrightarrow^{\partial} P)$ then

$$1 \to \pi_1\mathcal{M} \to C \longrightarrow^{\partial} P \to \pi_0\mathcal{M} \to 1$$

is exact. The results of Breen's section 6 (pp. 438-454) give an interpretation of the elements of $H^1(\mathcal{M})$ in terms of *gr*-fields *(gr-champ)* and torsors over them. We will not repeat that interpretation here.

The main point to note is that in $[T, \overline{W}G]$, the homotopies are not filtered. This gives the difference between $\Lambda_G(X)$ and $H^1(X, \mathcal{M})$, in general. Of course, if $P = 1$ so $\mathcal{M} = C[1]$ and C is abelian, then the filtration plays no rôle, so one does get $\Lambda_G(X)$ is the same as $H^1(X, \mathcal{M}) \cong H^2(X, C)$ as we saw. It thus will be necessary to analyse filtered homotopies more closely.

3.4 The structure of $\overline{W}G$.

As $\overline{W}G$ is the codomain of the filtered homotopies, it will be necessary to analyse the 'filling' structure of this simplicial set. As always, we assume that $\mathcal{M} = (C \to^{\partial} P)$ is a finite crossed module and that G is the associated simplicial group with $G_0 = P$, $G_1 = G \rtimes P$, etc.

Using the description of the $\overline{W}$-functor that was given in section 1.1, we have :

- $(\overline{W}G)_0$ is a single point;
- $(\overline{W}G)_1 = P$, so any $(1,i)$-horn in $\overline{W}G$ (and there is only one of them) has $\sharp(P)$ different 1-simplices that will fill it;
- $(\overline{W}G)_2$ has as typical element $z = (h_1, h_0)$, where $h_1 = (c,p) \in G_1$, $h_0 \in P$,

and in general,

- $(\overline{W}G)_n$ has typical elements $(h_{n-1}, \ldots, h_0)$ where $h_i \in G_i$ and so has the form,
$$(c_i, c_{i-1}, \ldots, c_1, p)$$
in the multiple semidirect product.

Any $(2,i)$-horn in $\overline{W}G$ has $\sharp(C)$ different fillers. To see why, we will examine briefly the question of a $(2,1)$-horn, the other calculations are similar. We thus suppose that we are given a $(2,1)$-horn $(x_2, \_, x_0)$, where $x_0, x_2 \in (\overline{W}G)_1 = G_0 = P$, are such that $d_0x_2 = d_1x_0$, (this last condition is trivial since $(\overline{W}G)_0$ is a single point). We have to find a $z = (h_1, h_0)$, $h_1 = (c,p) \in G_1 = C \rtimes P$, $h_0 = p' \in G_0$ that will satisfy the equations
$$d_0z = x_0, \quad d_2z = x_2.$$
Expanding these using the definitions of the face maps in $\overline{W}G$, we obtain
$$d_0z = h_0 = x_0$$
$$d_2z = d_1h_1 = x_2$$
and using the description of the face maps in G in terms of $\mathcal{M}$,
$$d_1h_1 = d_1(c,p) = p,$$
i.e. we need $z = ((c, x_2), x_0)$ and there are no extra conditions to determine c more precisely, hence the result on the number of fillers of a $(2,1)$-horn. It is worth noting that $d_1z = \partial c.x_2.x_0$, so if one know the boundary of a 2-simplex, there are exactly $ker\partial$ different ways to choose the simplex. (This can also be derived from the fact that NG_2 is zero, and hence $ker\partial$ is the second homotopy group of G.)

For $n \geq 3$, and (n,i)-horn has a unique filler. The calculation is again routine. We illustrate it with a $(3,1)$-horn giving fewer details. Explicit formulae can be given for the fillers.

Suppose the horn is $(x_3, x_2, \_, x_0)$, $x_i = (h_i, g_i) \in \overline{W}G_2$ with $h_i = (c_i, p_i)$, $g_i = p'_i$. The horn conditions, $d_1x_0 = d_0x_2$, etc. yield
$$\begin{array}{rcl} d_0h_0.g_0 &=& g_2 \\ d_1h_0 &=& g_3 \\ d_1h_3 &=& d_1h_2 \end{array} \qquad \text{or} \qquad \begin{array}{rcl} \partial c_0.p_0.p'_0 &=& p'_2 \\ \partial p_0 &=& p'_3 \\ \partial p_3 &=& p_2 \end{array}$$
We want to find a $z = (k, h, g)$ satisfying
$$d_iz = x_i, \qquad i = 0, 2, 3.$$
We write $k = (c', c, p)$. As $d_0z = (h,g) = x_0$, we know h and g before we start, and only need to solve for k. The only solution is $p = p_3, c = c_3, c' = c_2c_3^{-1}$. The calculation is easy and quite amusing!

3.5 Filtered homotopies.

The aim of this section is to consider the converse of proposition 3.4. As before we filter $\overline{W}G$ by $\{\overline{W}sk_kG\}$. Suppose $\lambda, \lambda' : T \to \overline{W}G$ is a filtered homotopy, so that $h|(sk_nT) \times I$ factors through $\overline{W}sk_{n-1}G$ if $n > 0$.

This means that for G, as above, associated with a crossed module, $\mathcal{M}$, since $sk_1G = G$, the set of filtered homotopy classes is the set of relative homotopy classes, $[(T, sk_1T), (\overline{W}, \overline{W}(sk_0G))]$.

A homotopy of colorings is a coloring of $T \times I$. In a filtered homotopy, as above, the edges in $T_0 \times I$ may be non-trivially colored by elements of P, but our analysis shows that, as sk_0G is just the constant simplicial group with value P in all dimensions, $\overline{W}sk_0G$ has the set P in dimension 1, is $P \times P$ in dimension 2, hence any $(2, i)$-horn in $\overline{W}sk_0G$ has a *unique* filler and thus $h|sk_1T \times I$ is in fact, determined by the colours $\lambda|sk_1T$ and $h|sk_0T \times I$. Moreover $\overline{W}sk_1G = \overline{W}G$, so the filtered homotopy h effectively reduces to conjugation by a set of elements of P.

Remark.

This nicely links up this analysis to that given in Yetter's earlier paper, [20]. His idea was to show that if two colorings λ of T and λ' of T' determined conjugate maps from Π_1X to G (G a finite group in this case), then (his lemma 2.7, p 8 of [20]) there is a common subdivision T'' of T and T' and a coloring λ'' of T'' such that λ'' restricts to λ on T and to λ' on T'. Hence λ and λ' determine the same element of $\Lambda_G(X)$. His proof does not go into much detail but this is not serious as he limits the manifolds considered to be surfaces and there no difficulties arise. Moreover by restricting to base points in each component of X, he can replace Π_1T and Π_1T' by $\pi_1(T, c)$ and $\pi_1(T', c')$, the collection of fundamental *groups* of T and T' based at these points. Now by the classical edge-path groupoid description of $\pi_1(X, c)$, these groupoids are independent of the choice of triangulation, at least up to conjugation (since the choice of an isomorphism between them, in part, depends on the choice of a maximal forest and in part on a choice of simplicial approximation).

In our situation, no restriction on the dimension of the manifolds has been made as the use of general subdivisions, rather than just edge stellar ones, means that the combinatorial arguments used in the construction can be replaced by more generally applicable topological ones. The invariants we are considering, including Π_2T, only depend on the 3-skeleton of T and thus on the homotopy 2-type of X. (Warning: that dependence is not 'up to isomorphism' but 'up to quasi-isomorphism', see Loday [17] This suffices for the calculation, as we are looking at filtered homotopy classes of morphisms with codomain G.)

Proposition 3.7 *Suppose that T is an ordered triangulation of an orientable manifold X, that $\lambda, \lambda' : T \to \overline{W}G$ are G-colorings, where G is associated to a crossed module, $\mathcal{M}$, and that $h : \lambda \simeq \lambda'$ is a filtered homotopy between them, then*

$$g_T^X(\lambda) = g_T^X(\lambda').$$

□

As h is a filtered homotopy, there is a commutative diagram

$$\begin{array}{ccc} T \times I & \xrightarrow{h} & \overline{W}G \\ \uparrow & & \uparrow \\ sk_1T \times I & \xrightarrow[h_1]{} & \overline{W}sk_0G \end{array}$$

where the vertical maps are inclusions and h_1 is the restriction of h. The idea is that, as we are proving a partial converse of Proposition 3.4, we must use h to define a coloring of some joint subdivision of T, but h may have come from composing many more basic homotopies (as given to us by 3.4), so we need to decompose h slightly as a first step. By our previous analysis, the filtered homotopy h is determined by the elements $h < v >$, v a vertex of T.

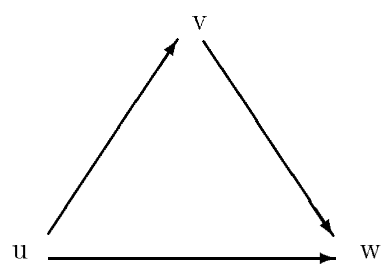
Consider the subdivision of an edge

$$v \longrightarrow w$$

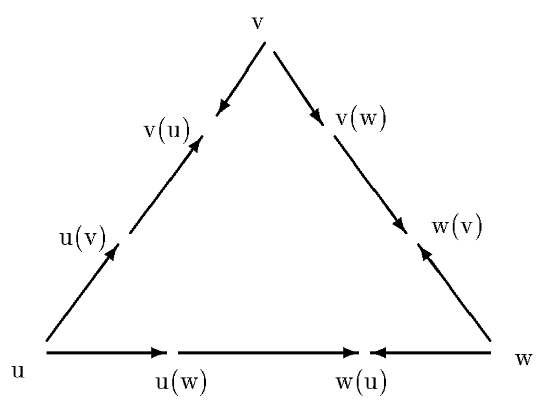
of T given by doubling each vertex:

$$v \longrightarrow v(w) \longrightarrow w(v) \longleftarrow w_0$$

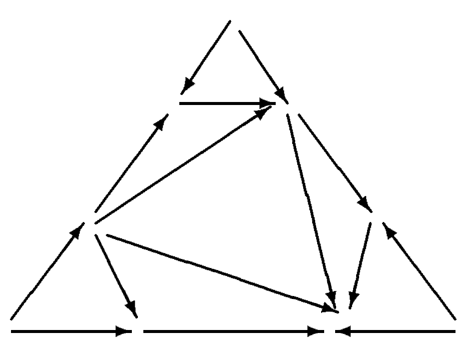
in each edge. Thus for a 2-simplex



we get an embryonic subdivision

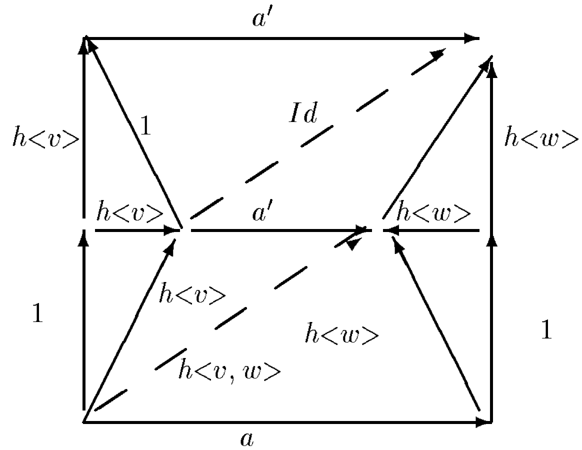


Each vertex is thus split to give as many new copies as there are edges incident to it (these have been labelled by the other end vertex). The ordering / orientation of the original simplex, then determines additional edges as shown



This propagates well to higher dimensions and ends up looking like a triangulation of one of the 'blow-ups' considered by Lawrence, [16]. The choice of this triangulation, T'' beyond the addition of the corners, is not made precise here as, in fact, the existence of a triangulation is almost all that is needed. The given one has the advantage that it allows an identifiable copy of the original simplex in its centre, but this is largely aesthetic, as we will see.

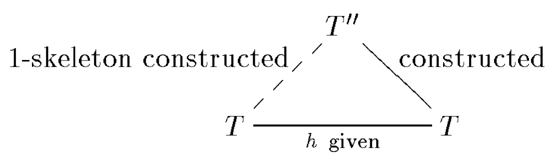
We next subdivide each square, $< v, w > \times I$ to add an additional vertex in the 1-simplices, $< v > \times I$ and $< w > \times I$, at which point we 'hang' the above triangulation:



This is then coloured as shown, where $a = \lambda < v, w >$ and $a' = \lambda' < v, w >$. It is clear that on the subdivision of $sk_1 T \times I$, this coloring is completely determined by the given data and that no conflicts arise.

Next fill the upper half of the diagram. This can be done in a unique way, so that each of the 'diamonds' is filled within $\overline{W} sk_0 G$. The resulting coloring, λ'' of the middle layer, T'', has a copy of $\lambda' < v_0, \ldots, v_n >$ in the middle of a 'halo' of degenerate material in the subdivision of the simplex corresponding to $< v_0, \ldots, v_n >$.

We thus have a 'prism' that can be visualised as



As $\overline{W} G$ is a Kan complex, we can extend over the remainder of the prism. The last 'face' then gives a way of mapping T'' into T making T'' a subdivision of T. The other face yields a way of considering T'' as a subdivision of a 'translated' copy of T obtained by mapping each vertex, v to some $v(w)$. We will denote this translated copy of T by T'. Note although the simplicial complex is the same the homeomorphism between its realisation and X is different. It is a *different* triangulation. On this triangulation T', λ' is still defined and

$$\mathrm{res}_{\mathrm{T}'',\mathrm{T}'}(\lambda'') = \lambda',$$

whilst

$$\mathrm{res}_{\mathrm{T}'',\mathrm{T}}(\lambda'') = \lambda.$$

This almost completes the proof except that we have

$$\lambda' : T' \to \overline{W}G$$

not

$$\lambda' : T \to \overline{W}G$$

as required. The map is the same, but within the colimit the actual triangulation is used not just the simplicial complex. To complete the proof, we have to show that

$$g_T^X(\lambda') = g_{T'}^X(\lambda'),$$

(where no apology is made for the ambiguity of the notation, which is to avoid excessive sub- or super-scripts). To prove this, just repeat the above argument with the identity filtered homotopy on λ'. This gives the same subdivision as above, and a coloring linking the original λ' with its translated version. □

Collecting up results we have

Theorem 3.8 *The vector space $Z_G(X)$ has a basis in bijection with the set*

$$[(GT, Gsk_1T), (G, sk_0G)]$$

of filtered homotopy classes of maps from $G(T)$ to G (or alternatively

$$[(T, sk_1T), (\overline{W}G, \overline{W}sk_0G)],$$

from T to $\overline{W}G$) for any triangulation T of X. □

It is clear that if T is a triangulation of X, then the filtered homotopy type of $G(T)$ should not depend on T, but merely on X. This can presumably be proved by comparing $G(T)$ with $G(Sing(X))$, and using the simplicial approximation thoerem, but the author has not checked this. The more restricted 'result' for $\Pi_2(T)$ is again 'clearly true', and would seem likely to be a consequence of the results of Brown and Higgins, [7], on subdivisions of CW-complexes and the effect they have on the crossed complex of the CW-structure. Again the author has not checked that this is so.

The above result implies:

Corollary 3.9 *The vector space $Z_G(X)$ has a basis in bijective correspondence with the set, $[(\Pi_2T, \Pi_1sk_1T), (G, sk_0G)]$, of filtered homotopy classes of maps from Π_2T to G.* □

Before passing to the interpretation in terms of bundles and cohomology, we note that this is true without restriction on the dimension of the manifold, however if X has a boundary, then it should be pointed out that the subdivision argument in 3.7 needs refining, as for a result 'rel boundary', the two colorings on the boundary should either agree, or, in more generality, should be handled first as part of an 'induction-up-the-skeleton' type argument. We will not be using such a 'rel boundary' result here, although clearly it could be of considerable interest.

3.5.1 Subdivision and G-bundles

Suppose G is associated with a finite crossed module, $\mathcal{M}$ so that if λ is a G-coloring of T, then $E(\lambda)$ has finite groupoid fibres. Because $\mathcal{M}$ is finite and we have complete knowledge of the fillers in $\overline{W}G$, the usual analysis of the correspondence between homotopy classes of classifying maps and isomorphism classes of bundles, in this case, gives to any specified homotopy between classifying maps, λ and λ', a specifiable isomorphism between the corresponding bundles. (The isomorphism is not claimed to be uniquely determined, only specifiable from the data on the homotopy, which is much weaker. The class of possible isomorphisms could be analysed, but this depends on the automorphisms of the fibre.) If $E(\lambda)$ and $E(\lambda')$ are the bundles corresponding to homotopic maps $\lambda, \lambda' : T \to \overline{W}G$, then the resulting isomorphism

$$\theta : E(\lambda) \to E(\lambda')$$

has the form $\theta(f,t) = (f \cdot \psi(t), t)$ (cf. May, [18] p.80 or Curtis, [10], p.160), where $\psi : T \to G$ is a mapping (not necessarily simplicial) and where, as $E(\lambda) = G \times_\lambda T$ etc., elements of $E(\lambda)$ are descibed as pairs, (f,t).

If H is a subsimplicial group of G, then θ would be called an H-equivalence if each $\psi(T)$ was in the subgroup, H. We actually have a slightly different situation and the following concept is introduced in an attempt to manage it efficiently.

Definition.

Suppose G is associated as always to $\mathcal{M} = (C, P, \partial)$ and that $E(\lambda)$, $E(\lambda')$ are isomorphic principal G-bundles. We will say that $E(\lambda)$ and $E(\lambda')$ are (G,P)-equivalent if there is an isomorphism

$$\theta : E(\lambda) \to E(\lambda')$$

given by : $\theta(f,t) = (f \cdot \psi(t), t)$ such that if $t \in sk_1 T$, then $\psi(t) \in sk_0 G$.

Proposition 3.10 *The vector space $Z_G(X)$ has a basis in bijective correspondence with the set of (G,P)-equivalence classes of principal G-bundles on T for any triangulation T of X.* □

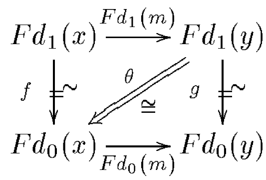
The proof should be clear from what has gone before. If more precision is required, an analysis of the proof of the usual form of the result that homotopic classifying maps give isomorphic bundles easily shows how to generalise to the filtered / relative form used here.

3.5.2 Subdivision and 2-descent data.

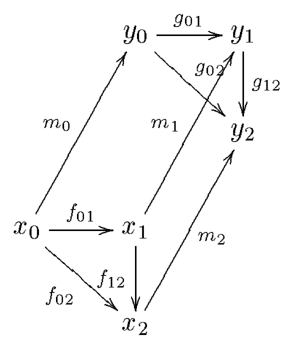
Recall from 2.4, that G-colorings of T correspond bijectively to normalised 2-descent data on the open star cover of T with coefficients in F, the constant sheaf with value the cat^1-group corresponding to G. Duskin, [11], takes $X_\bullet \to X$ to be a simplicial étale space over X and defines a 2-category $\underline{DES}^{(2)}(X_\bullet/X, F)$ with 0-cells the 2-descent data. We have postponed discussion of the 1-cells and 2-cells until now as they were not needed in the earlier discussion. A 0-cell was a triplet (α, f, x) where x was an object of FX_0 (so, as mentioned earlier, in our case, there is no choice here), f is a 1-cell in FX_1 of the form $f : Fd_1x \to Fd_0x$ (so corresponds to a $\lambda < v_0, v_1 >$ in the coloring) and finally α is a 2-cell in FX_2, $\alpha Fd_1(f) \to Fd_0(f)fd_2(f)$, (corresponding to the $\lambda < v_0, v_1, v_2 >^{-1}$ of Yetter's definition.

$\underline{\mathbf{DES}}^{(2)}(X_\bullet/X, F)$**(cf. Duskin, [11], p.257-260.)**

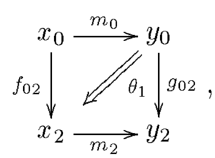
The morphisms of 1-cells of descent data in Duskin's sense are pairs, $(m, \theta) : (\alpha, f, x) \to (\beta, g, y)$, where m is an arrow (1-cell) of FX_0 of the form $m : x \to y$, and θ is a 2-cell of FX_1 of the form



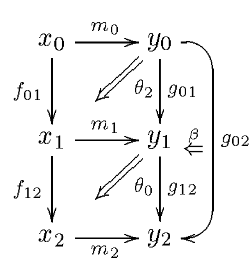
that is, $\theta : Fd_0(m) \cdot f \Rightarrow g \cdot Fd_1(m)$, subject to the conditions that
(a) θ is a isomorphism and
(b) the prism in FX_2



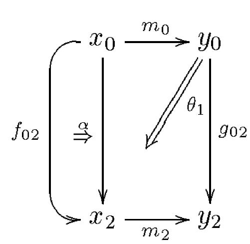
with 2-cells in each face so that, for instance, the backface is



obtained as the image under Fd_0, Fd_1 and Fd_2, is commutative in the 2-categorical sense, i.e. it yields two composable subdiagrams



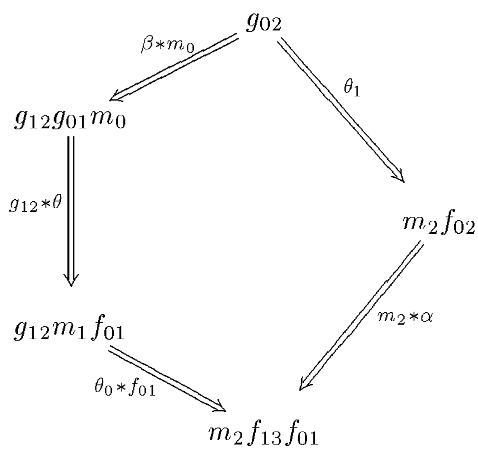
and



that, in turn, give composites that are required to be equal. Explicitly (cf. Duskin, [11], p.260) this yields

$$[Fd_0Fd_0(m) * \alpha] \cdot [Fd_1(\theta)] = [fd_0(\theta) * Fd_2(f)] \cdot [fd_0(g) * Fd_2(\theta)] \cdot [\beta * Fd_2Fd_1(m)]$$

This gives a commutative pentagon of 2-cells



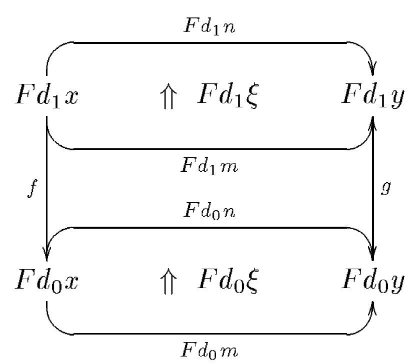
Finally a 2-cell or transformation of morphisms of descent data of $\underline{DES}^{(2)}(X_\bullet/X, F)$ with source (m,θ) and target (n,ρ), drawn

$$(\alpha, f, x) \underset{(n,\rho)}{\overset{(m,\theta)}{\underset{\Downarrow\xi}{\rightrightarrows}}} (\beta, g, y)$$

is a 2-cell ξ of FX_0 of the form

$$x \underset{n}{\overset{m}{\underset{\Downarrow\xi}{\rightrightarrows}}} y$$

such that the cylinder in



(with 2-cells θ and ρ in the back and front faces respectively) is commutative, in other words the square of 2-cells

$$\begin{array}{ccc} gFd_1m & \overset{g*Fd_1\xi}{\Longrightarrow} & gFd_1n \\ {\scriptstyle\theta}\Downarrow & & \Downarrow{\scriptstyle\rho} \\ gFd_0mf & \overset{g*Fd_0\xi}{\Longrightarrow} & gFd_0nf \end{array}$$

in FX_1 is commutative:

$$\theta \cdot (Fd_0\xi * f) = \rho \cdot (g * Fd_1\xi).$$

In our situation, all the x,y are the unique object of the 2-group, G, so each $m : x \to y$ is an element of P, the base group of the crossed modules, hence the defining property of θ is

$$\begin{array}{ccc} \cdot & \xrightarrow{m_0} & \cdot \\ {\scriptstyle \lambda<v_0,v_1>}\downarrow & \swarrow\!\!\!\swarrow_{\theta} & \downarrow{\scriptstyle \lambda'<v_0,v_1>} \\ \cdot & \xrightarrow[m_1]{} & \cdot \end{array}$$

with $Fd_1(m) = m_0$, $Fd_0(m) = m_1$, $f = \lambda < v_0, v_1 >$, $g = \lambda' < v_0, v_1 >$. This is a homotopy defined on the 1-skeleton of T. As $\theta \in G_1 \cong C \rtimes P$, it has the form (c, p) in general, (but we will need to restrict it to be of form $(1, p)$ for the filtered homotopy we will be considering). Similarly the prism condition again interprets as a homotopy, although there is no 3-cell to occupy the middle of the prism except for the identity.

Duskin, [11] p. 269, suggests a definition of a Čech cohomology with coefficients in a presheaf of 2-groups. Relative to a given (hyper)covering $X_\bullet \to X$, this is

$$\check{H}(X_\bullet/X, G) = \pi_0\underline{SS}(X_\bullet/X, \overline{W}G).$$

To get to an invariant, one, of course, passes to the colimit (but if X is a manifold or, in fact, even a polyhedron, the direct system stabilises, so no colimit is necessary for fine enough covers). Here $\underline{SS}$ is the simplicial set of morphisms and π_0 is, as usual, the connected component functor.

Definition

Suppose $X_\bullet \to X$ is a covering of X (e.g. defined by open star covers of some triangulation T) and F is the constant sheaf on X with value a (finite) cat^1-group, G (determined by $\mathcal{M} = (C, P, \partial)$), then the skeletal Čech cohomology of X relative to $X_\bullet \to X$ with coefficients in F (or G) will be

$$\check{H}(X_\bullet/X, (G, P)) \overset{\text{def}}{=} [(X_\bullet, sk_1X_\bullet)/X, (\overline{W}G, \overline{W}sk_0G)].$$

From our earlier results, it is clear that this is independent of $X_\bullet \to X$ provided that $X_\bullet$ is an open star covering of a triangulation (and, of course, we note that such coverings are cofinal amongst all coverings).

Proposition 3.11 *The vector space $Z_G(X)$ has a basis in bijective correspondence with the skeletal Čech cohomology of X with coefficients in G.* □

This skeletal Čech cohomology is also the set of connected components of a simplicial set given by the filtered maps, and finally is the set of connected component classes of 0-cells of a 2-category of filtered 2-descent data. This raises the possibility of 'enriching' the TQFT over simplicial vector spaces or vector 2-categories as a natural consequence of these interpretations. It is hoped to explore this in another paper, but at the time of writing certain extra facets of such a study have not been investigated yet.

4 Cobordisms.

The interpretations that have been given above for the spaces $Z_G(X)$ can be extended to handle cobordisms linking X and Y etc. Here there is no great advantage in dealing with the interpretations as they do not initially give much more insight than already available from Yetter's original treatment. It would seem that Yetter's description of the 'vacuum-to-vacuum' invariant $Z_G(X,)$ in [20], should have an analogue in terms of representations of $\Pi_2 X$ in G, but no clear description of this has yet emerged. Similarly the interpretation in terms of 'non-abelian' cocycles or descent data in the case when $\mathcal{M} = (A \to 1)$ with A abelian, yields exact sequences which are clearly linked with Yetter's 'vacuum-to-vacuum' invariant in this case, however the connections between the two insights are still some way from being obvious.

5 Conclusions

The realisation that Yetter's G-colorings have a neat simplicial description is important not because those models of TQFTs are by themselves important, but as Yetter mentions, they do raise interesting queries. The interpretations do suggest links with non-abelian cohomology and with stacks, gerbes, etc., even if in this case these are present in the very 'strict' form of simplicial fibre bundles. The methods suggest that Yetter's third question in [21], p. 123, can be answered in the positive. Any finite simplicial model of an n-type such as a n-truncated hypercrossed complex, [9], should yield a TQFT; the problem of the weightings is probably combinatorially tricky, but not impossible. (Yetter's comment about his Lemma 4 is avoided by using the Moore complex construction and a close analysis of the models of n-types.) More exactly, it seems likely that the strict group law of a cat^1-groupoid can be laxified to a monoidal category with enough extra structure - cf. well known results on simplicial monoids and group completions - and still yield $\overline{W}G$ so that the constructions go through. Here the problem of a left adjoint to $\overline{W}$ would be exacting if not impossibly difficult! The idea of 'fibre bundles' with finite hypergroupoids as fibres is again very challenging, as it links in with Grothendieck's 'Pursuing Stacks' programme.

The possibility of utilising simplicial enrichment is also interesting as it would result in simplicial vector spaces instead of simply vector spaces as the objects. The exact way to handle the weightings is however not at all clear.

Finally the description of elements of $Z_G(T)$ as being formal weighted sums of maps from $G(T)$ to G means that they can equally well described as being simplicial Hopf algebra maps from $\mathbb{C}[G(T)]$ to $\mathbb{C}[G]$. This then suggests replacing the simplicial Hopf algebra, $\mathbb{C}[G]$, by a more general simplicial Hopf algebra and rerunning the machinery.

References

[1] M.F.Atiyah, Topological Quantum Field Theories, Pub. Math. I.H.E.S., 68 (1988), 175-186.

[2] L.Breen, Bitorseurs et cohomologie non-abélienne, in *Grothendieck Festschrift I*, Progress in Mathematics 86 (1990), pp. 401-476.

[3] L.Breen, Théorie de Schreier supérieure, Ann. Scient. Ec. Norm. Sup., 25 (1992) pp.465-514.

[4] L.Breen, On the classification of 2-gerbes and 2-stacks, Astérisque, No. 225, 1994.

[5] R.Brown, Higher dimensional group theory, in *Low Dimensional Topology, Bangor, 1979*, ed. R.Brown and T.L.Thickstun, London Mathematical Society Lecture Notes Series 48, Cambridge University Press, 1982, pp. 215-238.

[6] R.Brown and P.J.Higgins, Colimit Theorems for relative Homotopy Groups, Journal Pure Applied Algebra, 22 (1981) pp.11-41.

[7] R.Brown and P.J.Higgins, Tensor products and homotopies for ω-groupoids and crossed complexes, Journal Pure Applied Algebra, 47 (1987), pp.1-33.

[8] R.Brown and J. Huebschmann, Identities among Relations, in *Low Dimensional Topology, Bangor, 1979*, ed. R.Brown and T.L.Thickstun, London Mathematical Society Lecture Notes Series 48, Cambridge University Press, 1982, pp.153-202.

[9] P.Carrasco and A.M.Cegarra, Group theoretic algebraic models for homotopy types, Journal Pure Applied Algebra 75 (1991) pp.195-235.

[10] E.B. Curtis, Simplicial Homotopy Theory, Advances Math. 6 (1971) pp.107-209.

[11] J.Duskin, An outline of a theory of higher dimensional descent, Bull. de la Soc. Math. de Belgique (série A) 41 (1989), pp. 249-277.

[12] W.G.Dwyer and D.M.Kan, Homotopy Theory and Simplicial Groupoids, Proc. Konink. Nederlandse Akad. van Wetenschappen A. 87 (1984), pp. 379-385.

[13] P.J.Ehlers, Simplicial Groupoids as Models for Homotopy Types, MSc. Diss. 1991, University of Wales, Bangor.

[14] P.G.Glenn, Realization of Cohomology Classes in Arbitrary Exact Categories, Journal Pure Applied Algebra, 25 (1982), pp.33-105.

[15] L.Illusie, Complexe cotangent et déformations, I, II, Lecture Notes in Mathematics, 239, 283, Springer-Verlag, (1971,1972).

[16] R.J.Lawrence, Triangulations, Categories and Extended Topological Field Theories, in *Conference proceedings on Quantum Topolgy* ed. R.Baadhio and L.H.Kaufmann, World Scientific, 1993.

[17] J.-L. Loday, Spaces with finitely many non-trivial homotopy groups, Journal Pure Applied Algebra, 24 (1982), pp.172-202.

[18] J.P. May, Simplicial Objects in Algebraic Topology, van Nostrand Math. Studies 11, 1967.

[19] E.H.Spanier, Algebraic Topology, McGraw-Hill series in Higher Mathematics, McGraw-Hill, 1966.

[20] D.N.Yetter, Topological Quantum Field Theories Associated to Finite Groups and Crossed G-sets, Journal of Knot Theory and its Ramifications, 1 (1992) pp.1-20.

[21] D.N.Yetter, TQFT's from Homotopy 2-types, Journal of Knot Theory and its Ramifications, 2 (1993) pp.113 - 123.