TQFTs from Homotopy n-types

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Abstract: Using simplicial methods developed in [22], we construct topological quantum field theories using an algebraic model of a homotopy n-type as initial data, generalising a construction of Yetter in [23] for n=1 and in [24] for n=2

Introduction

In [23], Yetter showed how to construct a topological quantum field theory with coefficients in a finite group. In [24], he showed that his construction could be extended to handle coefficients in a finite categorical group, or cat^1-group. These objects are algebraic models for certain homotopy 2-types. The topological quantum field theories thus constructed are (2+1) TQFTs, but the methods used do not depend on the manifolds being surfaces, except to avoid possible irregularities related to problems of triangulations in low dimensions.

Yetter ended that second note with some open questions, the third of which was: *can one carry out the same sort of construction for algebraic models of higher homotopy types?* In this note we will show that a fairly straightforward extension of Yetter's techniques does allow the construction of TQFTs with algebraic models of homotopy n-types as coefficients. We also will indicate to a limited extent some possible geometric interpretations of the basis elements of the Hilbert spaces involved.

The two keys to the extension are the preliminary work on simplicial descriptions of Yetter's constructions given by the author in [22] and earlier work by numerous researchers (Dwyer and Kan, [15], Loday, [19], Conduché, [12], Carrasco and Cegarra, [11] and others of the Granada algebra group, together with Jack Duskin, Don von Osdol and Ieke Moerdijk, and others), who have studied the problem of finding algebraic models for homotopy n-types from many different angles.

1 Algebraic Models of n-types

The modelling of n-types that we will consider will use simplicial groupoid methods. If one wants to start with general spaces, one uses some variant of the singular complex functor to get first to the category, *Simp*, of simplicial sets. As the spaces, we need are all triangulable, and triangulations are nearer the constructions (e.g. lattice integrals, etc.) of mathematical physics, we will use triangulations of a (compact) manifold as the starting point. (These are also finite beasties, so avoid some of the difficulties in proving finite dimensionality of the resulting Hilbert spaces later on.) Any triangulation yields a simplicial complex and thus a simplicial set.

From simplicial sets, one can go via the loop groupoid functor, G, of Dwyer and Kan, [15], to the category of simplicial groupoids. Here the objects of most interest to us will be the simplicially enriched groupoids, that is simplicial groupoids whose simplicial set of objects is constant. These are not only simplicial objects in the category of groupoids, but are also $Simp$-groupoids, i.e. groupoids enriched in the cartesian closed category, $Simp$, of simplicial sets. Thus these gadgets can be thought of not only as simplicial objects,

$$G_n \rightrightarrows G_{n-1} \qquad \cdots \qquad G_1 \rightrightarrows G_0$$

where each G_i is a groupoid with set of objects O and each face and degeneracy map is the identity on O, but also as groupoids, G in which as well as the set $G(x,y)$ of arrows in G from x to y for each x and y in the object set of G, there is a simplicial set $\mathbf{G}(x,y)_\bullet$, of 'higher dimensional arrows' such that the set of '0-dimensional arrows', $\mathbf{G}(x,y)_0$ is exactly $G(x,y)$ and the composition

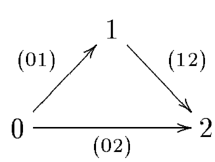
$$\begin{array}{rr} G(x,y)\times G(y,z)\to G(x,z) & \text{for } x,y,z,\in O = Ob(G) \\ \text{identities, } 1\to G(x,x) & \text{for } x,y\in O \end{array}$$

and inverse, $\qquad G(x,y)\to G(y,x) \qquad \text{for } x,y\in O$

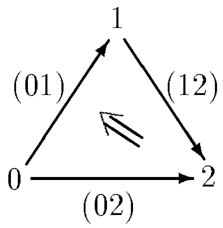
extend to give simplicial maps

$$\mathbf{G}(x,y)\times \mathbf{G}(y,z)\to \mathbf{G}(x,z), \text{ etc.}$$

In the simple case, where the set, O, is reduced to a singleton, this is 'just' a simplicial group) We refer the reader to the earlier paper [22] for a precise description of the loop groupoid, G. The idea behind that construction is : given a simplicial set, K, twist around the composite face maps so that a simplex σ in K_n can be thought of as having definite source and target vertices. For instance in the standard 2-simplex, $\Delta[2]$,



The source vertex of the top dimensional simplex, $(0,1,2)$ is the vertex 0 and the target is 1. The 'objects' of the simplicial groupoid, $G(\Delta[2])$ will be the vertices, $0,1,2$, of $\Delta[2]$, the 0-simplices of $G(\Delta[2])$ form the free groupoid on the 1-simplices of $\Delta[2]$. That much is easy. We now consider $(0,1,2)$ to be a 1-simplex $(\overline{0,1,2})$ in $G(\Delta[2])$ going from 0 to 1. Its faces are $d_1(\overline{0,1,2}) = (\overline{01})$ whilst $d_0(\overline{0,1,2}) = (\overline{02})(\overline{12})^{-1}$:



The functor $G : Simp \to SimpGpds$, has a right adjoint, $\overline{W}$, the classifying space functor, whose construction extends the classical one defined on simplicial groups. Morphisms into $\overline{W}G$ for G a simplicial group correspond to principal G-bundles by pulling back a universal such bundle over $\overline{W}G$ (see May, [20]).

Given any simplicial groupoid, G, its Moore complex, NG is given by

$$NG_n = \bigcap_{i=1}^{n} ker\{d_i : G_n \to G_{n-1}\}$$

with differential $\partial : NG_n \to NG_{n-1}$ being the restriction of d_0. If $n \geq 1$, this is just a disjoint union of groups, one for each object in the object set of G. If we write $G\{x\}$ for the simplicial group of elements that start and end at $x \in O$, then at object x, one has

$$NG\{x\}_n = (NG_n)\{x\}.$$

In dimension 0, however, one has $NG_0 = G_0$. Thus the $NG_n\{x\}$ are linked by the actions of the 0-simplices, acting by conjugation via repeated degeneracies.

The total structure, (NG, ∂), carries the structure of a hypercrossed complex of groupoids, that is a family of hypercrossed complexes in the sense of Carrasco and Cegara, [11], one for each $x \in O$, linked by the action of G_0.. The extension is easy to make. In that source, the notion of n-truncated hypercrossed complexes is discussed. This corresponds to one model of n-types, namely Moore complexes that vanish above dimension n.

Note that if G is a simplicial groupoid, and we define $t_{n]}G$ to be G in dimensions $< n$, to be $G_n/d_0(NG_{n+1})$ in dimension n, and then to be generated by this in higher dimensions with each NG_{n+r} killed off, then one gets a n-truncated hypercrossed complex for $N(t_{n]}G)$.

If G is a simplicial groupoid and H is a simplicial groupoid with NH_m trivial for $m > n$, then any morphism from G to H factors through $t_{n]}G$. The underlying simplicial set of $t_{n]}G$ is an n-hypergroupoid in the sense of Duskin and Glenn [17].

Suppose, as above, that NH_m is trivial for $m > n$.

If $n = 0$, then NH_0 is just the group, H_0.

If $n = 1$, then $\partial : NH_1 \to NH_0$ has a natural crossed module structure.

If $n = 2$, then

$$NH_2 \xrightarrow{\partial} NH_1 \xrightarrow{\partial} NH_0$$

is a 2-crossed module almost in the sense of Conduché, [12], but with the obvious extension from groups to groupoids.

Other algebraic models can be used, for instance, cat^{n-1}-groups, or crossed (n-1)-cubes, for which a useful reference is [21], however we will stick here with n-truncated hypercrossed complexes. (Note, since $G(T)$ is a 'loop' construction, the n-truncated hypercrossed complex of $G(T)$ models the (n+1)-type of T.) Strangely enough we will not need a detailed description of a hypercrossed complex structure. We will only need that

(i) hypercrossed complexes are chain complexes with extra structure;
(ii) the Moore complex functor factors through the forgetful functor from hypercrossed complexes to chain complexes;
(ii) a Dold-Kan theorem holds for hypercrossed complexes, i.e. from a hypercrossed complex, one can construct a simplicial group(oid) whose Moore complex is isomorphic to the given hypercrossed complex;
and finally,
(iv) all homotopy n-types can be modelled by hypercrossed complexes of length $n-1$ (i.e. with at most C_m, $0 \leq m < n-1$, non-trivial).

We will need finite hypercrossed complexes. We will abreviate this to FHCC with HCC used for the general case, and for most of the time, we will only need the group case, not the more general groupoid based case.

2 G-colorings of a triangulation.

We will work exclusively with PL-manifolds. This is only to be sure that the theory slots in nicely with what goes before. The technicalities of the constructions do not seem to depend on this restriction to any great extent. This is a disadvantage with these TQFTs as a compatibility with known or given structure would be desirable. We will assume, for simplicity, that the manifolds are d-dimensional, compact, oriented and without boundary, and that cobordisms are d+1 dimensional, PL, oriented, etc. The triangulations used will be ordered, again for simplicity.

A triangulation $\mathbf{T}$ of X is an abstract simplicial complex T and a homeomorphism $\phi : |T| \to X$. We therefore can think of the vertices of T as being elements of X, but while this is usually a harmless 'fudge', it is sometimes risky, and so needs to be kept in mind, particularly if ϕ and ϕ' are two different homeomorphisms that are 'near' to each other, say in a metric on X.

Given any two triangulations $\mathbf{T}$, $\mathbf{T}'$ of X, $\mathbf{T}'$ is a subdivision of $\mathbf{T}$ if there is a piecewise linear map $i : |T'| \to |T|$ such that
(a) $\phi i = \phi'$
(b) each simplex, σ, of $|T'|$ is contained in a simplex of $|T|$ of dimension no less than $\dim(\sigma)$.

(Much in this definition is, of course, redundant, but is included to remind the reader of some 'useful' information.) If both $\mathbf{T}$ and $\mathbf{T}'$ are ordered triangulations, then the induced map from the set of vertices of T to that of T' is assumed to be monotone. We will usually follow Yetter, [24], in assuming that subdivisions are obtained by iteratively starring at a point interior to an edge of a triangulation. This simplifies the arguments later on, but does not seem necessary other than for that simplification. (It avoids a lengthy case by case analysis and a multitude of different formulae.)

We assume given a finite $(n-1)$-truncated hypercrossed complex, NG, corresponding to a finite simplicial group G with NG_m trivial for $m \geq n$.

Definition

A coloring of T with values in G is a morphism

$$\lambda : G(T) \to G$$

of simplicial groupoids, or equivalently, a simplicial map

$$\lambda : T \to \overline{W}G.$$

We write $\Lambda_G(T)$ for the set of such G-colorings of T and $Z_G(X,T)$ for the vector space with basis, $\Lambda_G(T)$.

Proposition 2.1 *If G is a group (i.e. $n=1$) or a categorical group (i.e. $n=2$), then this notion of G-coloring is equivalent to that defined by Yetter.*

Proof: see [22]. □

If T' is obtained from T by iterated edge subdivision, than there is a morphism $\mathbf{r}^T_{T'}$: $G(T) \to G(T')$ of simplicial groupoids, obtained by realising the subdivision as a cone on a boundary and then filling algebraically, see [22]. This induces a function:

$$res_{T',T} : \Lambda_G(T') \to \Lambda_G(T)$$

as in the cases considered by Yetter,

Suppose now that (X,T) and (Y,S) are triangulated d-manifolds and $(M,\mathcal{T})$ is a triangulated cobordism with $\mathcal{T}$ extending T and S. Define $Z^!_G(M,\mathcal{T}) : Z_G(X,T) \to Z_G(Y,S)$ by

$$Z^!_G(M,\mathcal{T})(\lambda) = \sum\{\mu|_S \ : \ \mu \in \Lambda_G(\mathcal{T}),\ \mu|_T = \lambda\}.$$

This linear transformation is not independent of $\mathcal{T}$, for suppose we subdivide one edge, e of $\mathcal{T}$ without touching T or S, then writing $g_i = \sharp NG_i$, $i = 0,1,\ldots,n-1$, we have

Lemma 2.2 *For any given coloring μ of $\mathcal{T}$, fixed to be λ on T and λ' on S, there are*

$$g_0 g_1^{s_2(e)} g_2^{s_3(e)} \ldots g_d^{s_{d+1}(e)}$$

colorings of $\mathcal{T}'$ restricting to μ, where $s_k(e)$ is the number of k-simplices of $\mathcal{T}$ incident with e.

Proof.
We first note that writing

$$NG_i^{(r)} = \bigcap_{i=0, i\neq r}^{n} ker d_i,$$

then $\sharp(NG_i^{(r)}) = g_i$, since it is well known that $NG_i^{(r)}$ is in one-one correspondence with NG_i for any i.

In $\mathcal{T}'$ we have that e is subdivided to give e_0 and e_1, and $\mu(e) \in G_0$. We can choose $\mu'(e_0)$ in g_0 different ways and solve $\mu(e) = \mu'(e_0)\mu'(e_1)$ for $\mu'(e_1)$ to complete the definition of μ' on the 1-skeleton of T. Now for each 2-simplex, σ, of $\mathcal{T}$ incident to e, we fill a $(2,0)$-horn in $\overline{W}G$ to obtain μ' on the half simplex containing e_0. The value on the other half is determined by the fact that the value $\mu(\sigma)$ is given. The choice of fillers can be done in $\sharp(NG_1)$-ways, i.e. in g_1 different ways and so on. At each stage, there are g_k choices to be made on each of $s_{k+1}(e)$ $(k+1)$-simplices. This proves the lemma. □

Remark.

The above proof relies implicitly on the filling structure of $\overline{W}G$ for G a finite simplicial group. In low dimensions, i.e. for $n=2$, this was explicitly used in [22]. That analysis extends in the obvious way to higher dimensions using Conduché's remark, [12], that the Dold-Kan decomposition of the group of n-simplices in a simplicial *abelian* group has an easy extension that gives an analogous *semi-direct* product decomposition for the group of simplices of an arbitrary simplicial group. (This result is central to both the hypercrossed complex approach of Carrasco and Cegarra, [11] and to my own algebraic version of Loday's theorem on n-types, [21].)

Now suppose M is an arbitrary manifold, possibly with boundary ∂M, and let $\mathcal{T}$ be a triangulation of M. For each $k \geq 0$, let $sk_k\mathcal{T}$ be the k-skeleton of $\mathcal{T}$ and write $\chi_k(\mathcal{T}) = (-1)^k\chi(sk_k\mathcal{T})$, where χ is the Euler characteristic function. Further let $\chi_k^{int}(\mathcal{T}) = \chi_k(\mathcal{T} \cap (M \setminus \partial M))$ be the contribution given by the interior simplices of $\mathcal{T}$ and $\chi_k^{\partial}(\mathcal{T})$ that given by the boundary simplices.

Lemma 2.3 *Suppose that $\mathcal{T}'$ is obtained from $\mathcal{T}$ by subdividing the edge e in the interior of M, then for each k*

$$\chi_k^{int}(\mathcal{T}') = \chi_k^{int}(\mathcal{T}) + s_{k+1}(e)$$

where as before $s_{k+1}(e)$ is the number of $(k+1)$-simplices of $\mathcal{T}$ incident to e.

Proof

If K is a simplicial complex, then subdividing at an edge in K does not change the Euler characteristic of K, which is invariant under subdivision, however $sk_k(\mathcal{T})$ and $sk_k(\mathcal{T}')$ differ by having <u>new</u> k-simplices, dividing in two each of the $s_{k+1}(e)$-simplices of dimension $k+1$ that are incident to e. □

Returning to our cobordism M joining X to Y, define

$$Z_G^!(M,T,S) = \prod g_k^{-\chi_k^{int}(\mathcal{T})} Z_G^!(M,\mathcal{T})$$

Proposition 2.4 *The map $Z_G^!(M,T,S)$ is independent of the choice of triangulation, $\mathcal{T}$, extending T and S to the cobordism.*

Proof

Subdivide $\mathcal{T}$ at an interior edge. Clamping the colorings on T and S, by 2.2 and 2.3, this increases the number of colorings by the same factor as we have divided by above. □

Finally we need to adjust this linear transformation so as to make it compose well. Suppose M is a cobordism from X to Y and N is one from Y to Z. Let T, S, R be triangulations of X, Y, Z respectively. As in the simple cases considered by Yetter, there is an anomaly between $Z_G^!(N,S,R)Z_G^!(M,T,S)$ and $Z_G^!(M +_Y N, T, R)$, since in the former the contribution of S is not free. Calculating we find

$$Z_G^!(N,S,R)Z_G^!(M,T,S) = \prod g_k^{\chi_k(S)} Z_G^!(M +_Y N, T, R)$$

so to adjust the linear transformations $Z_G^!(M,T,S)$ to allow for this factor, we assign half of this contribution to each end of the cobordism. This gives

$$Z_G(M,T,S) = \prod g_k^{-\frac{1}{2}\chi_k^{\partial}(\mathcal{T})} Z_G^!(M,T,S),$$

and now

$$Z_G(N,S,R)Z_G(M,T,S) = Z_G(M +_Y N, T, R).$$

Summing up we have

$$Z_G(M,T,S) = \prod g_k^{-(\chi_k^{int}(\mathcal{T}) + \frac{1}{2}\chi_k^{\partial}(\mathcal{T}))} Z_G^!(M,\mathcal{T})$$

and an analoque of Yetter's Proposition 10, [24], p.118.

Following the same plan as Yetter, we now adjust the $res_{T'T}$ maps to be compatible with cobordisms. If T' is a subdivision of T, both being triangulations of X, define $\mathrm{res}_{T',T}$: $Z_G(X,T') \to Z_G(X,T)$ by

$$\mathrm{res}_{T',T} = \prod g_k^{\frac{1}{2}(\chi_k(T')-\chi_k(T))} res_{T',T}.$$

Lemma 2.5 *Suppose M, X, Y, $\mathcal{T}$, T, T' and S are as before, then the diagram*

$$\begin{array}{ccc} Z_G(X,T') & \xrightarrow{Z_G(M,T',S)} & Z_G(Y,S) \\ {\scriptstyle \mathrm{res}_{T',T}}\downarrow & \nearrow{\scriptstyle Z_G(M,T,S)} & \\ Z_G(X,T) & & \end{array}$$

commutes.

Proof

The correcting factor applied to $res_{T',T}$ adjusts for the new simplices in each dimension and thus for the new colorings that have to be counted. □

There is a similar commutative diagram for S', an edge stellar subdivision of S.

The vector spaces $Z_G(X,T)$ and these adjusted restriction maps form a diagram of vector spaces over the category with triangulations of X as its objects and edge stellar subdivisions as maps. Define $Z_G(X) = colim(Z_G(X,T), \mathrm{res}_{T',T})$.

If M is a cobordism from X to Y, then the implication of lemma 2.5 is that M induces a natural transformation of the corresponding diagrams for X and Y and hence a natural transformation

$$Z_G(M) : Z_G(X) \to Z_G(Y).$$

If N is a cobordism from Y to Z then

$$Z_G(N)Z_G(M) = Z_G(M +_Y N),$$

so Z_G is a functor from *d-cobord*$_{PL}$ to the category of finite dimensional complex vector spaces. As there is a single coloring of the empty d-manifold, $Z_G(\emptyset) \cong \mathbb{C}$ and as $\Lambda_G(X \coprod Y) \cong \Lambda_G(X) \times \Lambda_G(Y)$, we also have $Z_G(X \coprod Y) \cong Z_G(X) \otimes Z_G(Y)$, so the functor Z_G is monoidal. We thus have

Theorem 2.6 *For any dimension d, the construction above applied to* d-cobord$_{PL}$*, the category of d-dimensional PL-manifolds, gives rise to a (d+1)-dimensional TQFT.* □

3 Interpretation

As in [22], there would seem to be at least three useful interpretations of these G-colorings and two different levels of such interpretation. The three are

(i) maps between truncated hypercrossed complexes (or equivalent algebraic models for homotopy types);

(ii) simplicial bundles with (n-1)-types as fibres;

and
(iii) descent data and non-abelian cocycles.

The two levels are (a) the colorings and (b) the basis elements of $Z_G(X)$. Here we will restrict to level (a) as I have not yet examined the equivalence relation that leads from $Z_G(X,T)$ to $Z_G(X)$ although presumably many of the arguments given in [22] will either apply directly or extend without too much bother. The first two of the interpretations will be handled in more detail than the third, since non-abelian descent data in higher dimensions is not available in the literature and here would not be the place to develop it in detail. The amount we would need would be small, but as here it would be in a very special case, it would seem better to attempt to develop it more fully elsewhere.

(i) **Morphisms of truncated HCCs**

Let G be, as before, a finite simplicial group with NG_m trivial for $m \geq n$, (so NG is a (n-1) truncated FHCC).

If X is a d-dimensional manifold and T is a triangulation of X, then we can form $G(T)$ as before and obtain colorings as morphisms

$$\lambda : G(T) \to G.$$

Passing to the Moore complexes, this gives:

$$N(\lambda) : NG(T) \to NG,$$

but as NG_m is trivial for $m \geq n$, this morphism of HCCs factors through $t_{n-1]}NG(T)$.

Unlike $\Pi_1 T$, which is the case $n = 2$ of this construction, $t_{n-1]}NG(T)$ is not a homotopy invariant as homotopies translate to algebraic homotopies between these and will not reduce to isomorphisms. In fact, these 'models' are extremely big and it is to be hoped that some progress will be made on finding 'minimal' models in a suitable sense. Some progress in this direction has been made by Baues, [2] and [3], especially in low dimensions.

Alternative models can be used such as the cat$^{(\mathrm{n}-1)}$-groups of Loday, [19], for which a van Kampen theorem aids computation, cf. Brown and Loday, [9] and [10]. Equivalent to that approach is one involving crossed n-cubes, eg. in [21]. This generalises more directly the crossed module approach for $n = 2$ mentioned in [22]. Finally if G is a group T-complex, then NG is a crossed complex, C, say and we can factor out other elements to replace $t_{n-1]}NG(T)$ by $\pi(T)$, the crossed complex of the skeletally filtered complex, T. This case may lead to interesting developments as then $\Lambda_G(T)$ is the zeroth level of the internal hom $CRS(\pi(T), C)$, which corresponds to an ω-groupoid, which will be truncated, since C is. Hence this will be a $(n+1)$-fold groupoid. (The basic references to this latter theory are in the work of Brown and Higgins, see [7], and [8]. The link with simplicial groupoids is explored in Ehlers and Porter, [16].) Taking a $\mathbb{C}$-vector space on each level of $CRS(\pi(T), C)$ would yield a small '$\mathbb{C}$-additive $(n+1)$-fold groupoid'. To my knowledge, no structure theories of such gadgets have been considered as yet.

(ii) **Simplicial bundles with (n-1)-types as fibres.**

We saw earlier that a G-coloring

$$\lambda : G(T) \to G$$

is adjoint to some

$$\overline{\lambda} : T \to \overline{W}G.$$

On $\overline{W}G$, there is a principal G-bundle (cf. May, [20], or Curtis, [13]), $WG \to \overline{W}G$, which has the underlying simplicial set of G as fibre. This simplicial set is an n-hypergroupoid in the sense of Glenn, [17]. The simplicial morphism $\overline{\lambda}$ induces a principal G-bundle on T. These simplicial fibre bundles, fibred in n-hypergroupoids, are again a new feature of this theory and nothing explicit is known about them. Whether or not they arise in a physical context is at present anyone's guess, but given that stacks of groupoids and similar gadgetry occur in the work of Brylinski and others, the question is not an idle one. It should also be mentioned that bundles of n-types are a (too rigid) form of the n-stacks so diligently pursued by Grothendieck [18].

(iii) **Descent data.**

The interpretation in terms of n-descent data is now 'clear'. It should follow the treatment for $n = 2$ given in [22], which was based on Duskin, [14] and Breen, [4], [5], and [6]. Given a finite $(n-1)$-truncated hypercrossed complex, C and G, a simplicial group with $NG \cong C$, as HCCs. We can define a constant presheaf of simplicial groupoids on X, which will also be denoted G.

Suppose that $\mathcal{C}_X$ is the category of spaces étale over the manifold X. The category $\mathcal{C}_X$ is naturally a site in the sense of Grothendieck and Verdier. Any open cover $\mathcal{U}$ (or more generally, any covering, $\mathcal{U} \to X$, in the sense of Grothendieck topologies) defines an augmented simplicial object in $\mathcal{C}_X$ by the usual nerve construction. This will be denoted $X(\mathcal{U})_. \to X$. In particular if $\mathbf{T}$ is a triangulation of X, $\mathcal{U}$ can be chosen to be the associated star open cover.

Now consider the simplicial set $\underline{Simp}(N(\mathcal{U}), \overline{W}(G))$, where $N(\mathcal{U})$ is the ordinary nerve of the cover $\mathcal{U}$. This will be isomorphic to T if $\mathcal{U}$ is the star open cover of $\mathbf{T}$. Duskin, [14], defines

$$\check{H}(X(\mathcal{U})./X, G) = \pi_0 \underline{Simp}(N(\mathcal{U}), \overline{W}(G))$$

in this sort of context. This cohomology set is relative to $\mathcal{U}$, but one can pass in the usual way to a colimit to get a set independent of $\mathcal{U}$.

As $N(\mathcal{U}) \cong T$, if $\mathcal{U}$ is the open star cover of $\mathbf{T}$, and, for varying $\mathbf{T}$, this happens cofinally with respect to the usual ordering on covers, we can link up this colimit set

$$Colim \check{H}(X(\mathcal{U})./X, G) = \pi_0 \underline{Simp}(N(\mathcal{U}), \overline{W}(G))$$

with the generalised form of Yetter's theory introduced above. G-colorings are interpreted as n-cocycles or n-descent data in this context. It should however be pointed out that the colimit process used in this Čech-style non-abelian cohomology and that used in section 2 to define $Z_G(X)$ are not the same although they are related (see [22] for a discussion when n=2).

As mentioned earlier, this area of non-abelian cohomology looks as if it needs a lot of detailed investigation to exploit its potential to the full.

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