Notes on higher dimensional groups and related topics

Christopher D Wensley

December 23, 2008

Contents

Preface

The aim of these notes is to collect together, using a common notation and right actions, information about crossed modules; cat1-groups; crossed squares; and related structures.

Inspiration has come from Ronnie Brown; Tim Porter; postgraduate students; and the many visitors to the department in Bangor.

Help in the preparation of these notes has initially been provided by Murat Alp and Robert Rodrigues. A seminar course for Gareth Evans and Richard Lewis during 2002/2003 resulted in many corrections and additions.

This is work in progress, so there are many comments in the text of the form "[**Such-and-such needs doing.**]".

The main sources used are:

- M. Alp and C.D. Wensley, *Enumeration of cat1-groups of low order*, Int J Alg Comp 10 (2000) 407–424.
- R. Brown and N.D. Gilbert, *Algebraic models of 3-types and automorphism structures for crossed modules*, Proc. London Math. Soc. (3) 59 (1989) 51–73.
- R. Brown and J. Huebschmann, *Identities among relations*, in Low-dimensional topology, London Math. Soc. Lecture Note Series 48, ed. R Brown and T L Thickstun, Cambridge University Press, 1982, pp.153–202.
- R. Brown and İ. İçen, *Homotopies and automorphisms of crossed modules of groupoids*, Applied Categorical Structures 11 (2003) 185–206.
- R. Brown and J.-L. Loday, *Van Kampen theorems for diagram of spaces*, Topology 26 (1987) 311–335.
- G.J. Ellis, *Crossed modules and their higher dimensional analogues*, Ph.D. Thesis, University of Wales, Bangor, 1984.
- G.J. Ellis and R. Steiner, *Higher-dimensional crossed modules and the homotopy groups of* $(n+1)$*-ads*, J Pure and Appl. Algebra 46 (1987) 117–136.
- K.H. Kamps and T. Porter, *2-groupoid enrichments in homotopy theory and algebra*, K-Theory 25 (2002) 373–409.
- K.J. Norrie, *Crossed modules and analogues of group theorems*, Ph.D. Thesis, King's College, London, January 1987.
- T. Porter, *n-types of simplicial groups and crossed n-cubes*, Topology 32 (1993) 5–24.
- (more to be added)

1 Crossed Modules and Cat^1-Groups

In this section we recall the descriptions of four equivalent categories: **XMod**, the category of crossed modules and their morphisms; **Cat**1, the category of cat^1-groups and their morphisms; **GpGpd**, the subcategory of groups in the category **Gpd**of groupoids; and **2-Gp**, a subcategory of **2-Cat**. We also describe functors between these categories which exhibit the equivalences.

1.1 Pre-crossed and Crossed Modules

Let Q and R be groups acting upon themselves by conjugation:

$$q_0{}^q = q^{-1}q_0q, \qquad r_0{}^r = r^{-1}r_0r.$$

A *pre-crossed module* $\mathcal{Q} = (\delta : Q \to R)$ consists of a group homomorphism δ, called the *boundary* of $\mathcal{Q}$, together with an action $\alpha : R \to \mathrm{Aut}(Q)$ such that δ is an R-morphism. So, for all $q \in Q$ and $r \in R$,

$$\textbf{X1:} \quad \delta(q^r) \quad = \quad (\delta q)^r \ .$$

An alternative notation is to say that (Q, δ) is a pre-crossed R-module.

The pre-crossed module $\mathcal{Q}$ is a *crossed module* if it also satisfies, for all $q_0, q \in Q$,

$$\textbf{X2:} \quad q_0{}^{\delta q} \quad = \quad q_0{}^q \ .$$

A *morphism of pre-crossed modules* $\alpha : \mathcal{Q}_1 \to \mathcal{Q}_2$ is a pair $\alpha = (\ddot{\alpha}, \dot{\alpha})$, where $\ddot{\alpha} : Q_1 \to Q_2$ and $\dot{\alpha} : R_1 \to R_2$ are homomorphisms satisfying

$$\delta_2\ddot{\alpha} = \dot{\alpha}\delta_1, \quad \ddot{\alpha}(q^r) = (\ddot{\alpha}q)^{\dot{\alpha}r},$$

making the following diagram commute:

$$\begin{array}{ccc} Q_1 & \xrightarrow{\ \ddot{\alpha}\ } & Q_2 \\ {\scriptstyle \delta_1}\downarrow & & \downarrow{\scriptstyle \delta_2} \\ R_1 & \xrightarrow[\ \dot{\alpha}\ ]{} & R_2 \end{array} \tag{1}$$

When $\mathcal{Q}_1, \mathcal{Q}_2$ are both crossed modules then α is a *morphism of crossed modules* without any further condition. We thus obtain the category **PreXMod**of pre-crossed modules and their morphisms, and the category **XMod**of crossed modules and their morphisms. Furthermore, **XMod**is a full subcategory of **PreXMod**.

When $\mathcal{Q}_2 = \mathcal{Q}_1$ and $\ddot{\alpha}, \dot{\alpha}$ are automorphisms then α is an automorphism of $\mathcal{Q}_1$. The group of automorphisms is denoted by $\mathrm{Aut}(\mathcal{Q}_1)$.

We have to be careful about a notational problem which arises because, although we are using right actions, we are still writing functions of the left. Thus the group of automorphisms of R has multiplication $\beta_1 * \beta_2$ given by

$$(\beta_1 * \beta_2)r \ = \ \beta_2(\beta_1(r)) \quad \text{or, more conveniently,} \quad \beta_2\beta_1 r \ ,$$

where β_1 is applied *first* to r, and β_2 *second*.

1.2 Examples of Crossed Modules

Standard constructions for crossed modules include the following:

1. A *conjugation crossed module* is an inclusion of a normal subgroup $S \trianglelefteq R$, where R acts on S by conjugation.

2. An *automorphism crossed module* has as range a subgroup R of the automorphism group $\mathrm{Aut}(S)$ of S which contains the inner automorphism group $\mathrm{Inn}(S)$ of S. The boundary maps $s \in S$ to the inner automorphism of S by s.

3. A *zero boundary crossed module* has an R-module as source and $\partial = 0$.

4. Any homomorphism $\partial : S \to R$, with S abelian and im ∂ in the centre of R, provides a crossed module with R acting trivially on S.

5. A *central extension crossed module* has as boundary a surjection $\partial : S \to R$ with central kernel, where $r \in R$ acts on S by conjugation with $\partial^{-1}r$.

6. The *direct product* of $\mathcal{X}_1 = (\partial_1 : S_1 \to R_1)$ and $\mathcal{X}_2 = (\partial_2 : S_2 \to R_2)$ is $\mathcal{X}_1 \times \mathcal{X}_2 = (\partial_1 \times \partial_2 : S_1 \times S_2 \to R_1 \times R_2)$ with R_1, R_2 acting trivially on $S_2,\ S_1$ respectively.

Here is a verification for example 5. Suppose $s_1, s_2 \in \partial^{-1}r$. Then $s_2 = s_1k$ for some $k \in \ker\partial$, so $s^{s_2} = s^{s_1k} = s_1^{-1}(k^{-1}sk)s_1 = s^{s_1}$, and the action is well-defined. The two axioms are then easily verified.

1.3 Properties of Crossed Modules

Lemma 1.1

(i) *The kernel K of ∂ is central in S, and so is abelian.*

(ii) *The image $J = \mathrm{im}\,\partial$ is normal in R, and so we may define $C = \mathrm{coker}\,\partial$, with natural map $\nu : R \to C$, and hence obtain an exact sequence of groups*

$$1 \longrightarrow J \xrightarrow{\ \iota\ } R \xrightarrow{\ \nu\ } C \longrightarrow 1\,.$$

(iii) *The group J acts trivially on the centre ZS of S, and so trivially on K. Hence K inherits an action of C, making K a C-module with $k^{Jr} := k^r$, and giving a crossed module $(\nu\partial|_K : K \to C)$.*

(iv) *If $\partial' : S \to J$ is the restriction of ∂, there is an exact sequence of R-groups*

$$1 \longrightarrow K \longrightarrow S \xrightarrow{\ \partial'\ } J \longrightarrow 1\,.$$

(v) *The image J acts trivially on the abelianisation $S^{\mathrm{ab}} = S/[S,S]$ of S, and so S^{ab} is also a C-module, with action*

$$([S,S]s)^{(Jr)} \;=\; [S,S](s^r)\;. \tag{2}$$

Proof:

(i) If $k \in K$ and $s \in S$ then, by **X2**,

$$s \;=\; s^{\partial k} \;=\; k^{-1}sk \qquad \text{and so} \qquad ks \;=\; sk\,.$$

(ii) This follows immediately from **X1**.

(iii) If $z \in ZS$ then, by **X2**, $z^{\partial s} = s^{-1}zs = z$.

(iv) The kernel of ∂' is K and ∂' is surjective.

(v)
$$[S,S]1 \;=\; [S,S][s_1, s^{-1}] \;=\; [S,S]\, s^{\partial s_1}\, s^{-1} \qquad \Rightarrow \qquad [S,S]s \;=\; [S,S]s^{\partial s_1}\,.$$

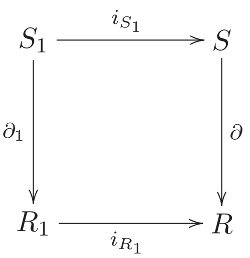
Hence J acts trivially on $S^{\rm ab}$, and it is easy to check that (2) defines an action.

□

1.4 Sub-crossed modules

Definition 1.2 *A crossed module $\mathcal{X}_1 = (\partial_1 : S_1 \to R_1)$ is a sub-crossed module of $\mathcal{X} = (\partial : X \to Y)$, written $\mathcal{X}_1 \leqslant \mathcal{X}$, if*

- *S_1, R_1 are subgroups of S, R respectively,*
- *∂_1 is the restriction of ∂ to S_1,*
- *the action of R_1 on S_1 is induced by the action of R on Q.*



Definition 1.3 *The* inclusion morphism $i = (i_{S_1}, i_{R_1}) : \mathcal{X}_1 \to \mathcal{X}$ *consists of the two subgroup inclusions $i_{S_1} : S_1 \to S$ and $i_{R_1} : R_1 \to R$.*

Definition 1.4 *The sub-crossed module $\mathcal{X}_1$ is a* normal sub-crossed module *of $\mathcal{X}$, written $\mathcal{X}_1 \trianglelefteq \mathcal{X}$, if*

(a) *$R_1 \trianglelefteq R$,*

(b) *${s_1}^r \in S_1$ for all $r \in R,\ s_1 \in S_1$,*

(c) *$s^{-1}s^{r_1} \in S_1$ for all $r_1 \in R_1,\ s \in S$.*

Note that these conditions imply that $S_1 \trianglelefteq S$, for ${s_1}^s = {s_1}^{\partial s}$ by **X2:** and belongs to S_1 by (b).

Proposition 1.5 [Is this true?] *Given two normal sub-crossed modules $\mathcal{X}_1, \mathcal{X}_2$ of $\mathcal{X}$, there is a third normal sub-crossed module of $\mathcal{X}$ called the* commutator sub-crossed module $[\mathcal{X}_1, \mathcal{X}_2]$, *having*

- *source group $[S_1, S_2]$,*
- *range group $[R_1, R_2]$,*
- *the restriction ∂' of ∂ to $[S_1, S_2]$ as boundary map.*

Proof: First note that $\partial'[s_1, s_2] = [\partial s_1, \partial s_2]$, and that $[s_1, s_2]^{r_1} = [s_1{}^{r_1}, s_2{}^{r_1}]$, so $[\mathcal{X}_1, \mathcal{X}_2]$ is a sub-crossed module of $\mathcal{X}$. To show normality we check that

(a) $[R_1, R_2] \trianglelefteq R$, since $r^{-1}[r_1, r_2]r = [r^{-1}r_1r, r^{-1}r_2r]$,

(b) $[s_1, s_2]^r = [s_1{}^r, s_2{}^r] \in [S_1, S_2]$,

(c) $s^{-1}s^{[r_1,r_2]} = (s^{-1}s^{r_1^{-1}})((s^{r_1^{-1}})^{-1}(s^{r_1^{-1}})^{r_2^{-1}})((s^{r_1^{-1}r_2^{-1}})^{-1}(s^{r_1^{-1}r_2^{-1}})^{r_1})((s^{r_1^{-1}r_2^{-1}r_1})^{-1}(s^{r_1^{-1}r_2^{-1}r_1})^{r_2})$.

[But (c) does not show that the left-hand side is in $[S_1, S_2]$.] □

1.5 Properties of morphisms of crossed modules

Let $\mu = (\ddot{\mu}, \dot{\mu}) : \mathcal{X} \to \mathcal{X}'$ be a morphism of crossed modules. The kernel of μ is the sub-crossed module

$$\ker \mu \ = \ (\partial\,|_{\ker \ddot{\mu}} \ \ : \ \ker \ddot{\mu} \to \ker \dot{\mu})$$

of $\mathcal{X}$, as shown in the following diagram.

$$\begin{array}{ccccc}
\ker \ddot{\mu} & \xrightarrow{\ i\ } & S & \xrightarrow{\ \ddot{\mu}\ } & S' \\
\Big\downarrow{\scriptstyle \partial|_{\ker \ddot{\mu}}} & & \Big\downarrow{\scriptstyle \partial} & & \Big\downarrow{\scriptstyle \partial'} \\
\ker \dot{\mu} & \xrightarrow[\ i\ ]{} & R & \xrightarrow[\ \dot{\mu}\ ]{} & R'
\end{array}$$

Lemma 1.6 *The kernel of μ is a normal sub-crossed module of $\mathcal{X}$.*

Proof:

(a) $\ker \dot{\mu} \trianglelefteq R$,

(b) if $\ddot{\mu}s = 1$ then $\ddot{\mu}(s^r) = (\ddot{\mu}s)^{\dot{\mu}r} = 1^{\dot{\mu}r} = 1$,

(c) if $\dot{\mu}r = 1$ then $\ddot{\mu}((s^{-1})^r s) = (\ddot{\mu}s^{-1})^1(\ddot{\mu}s) = 1$.

□

Theorem 1.7 *Given a normal sub-crossed module $\mathcal{X}_1$ of a crossed module $\mathcal{X}$, there is a quotient crossed module*

$$\mathcal{X}/\mathcal{X}_1 \ = \ (\delta : S/S_1 \to R/R_1)$$

where the action is defined by

$$(S_1 s)^{R_1 r} \ := \ S_1(s^r)$$

and the boundary map is given by

$$\delta(S_1 s) \ := \ R_1(\partial s).$$

Proof: We first check that we do have an action:

$$(S_1s)^{R_1r}(S_1s')^{R_1r} \ = \ (S_1(s^r))(S_1(s'^r)) \ = \ S_1(s^r)(s'^r) \ = \ S_1(ss')^r \ = \ (S_1(ss'))^{R_1r},$$
$$((S_1s)^{R_1q})^{R_1r} \ = \ (S_1(s^q))^{R_1r} \ = \ S_1(s^q)^r \ = \ S_1(s^{qr}) \ = \ (S_1s)^{R_1(qr)}.$$

Then we check the crossed module axioms:

M1: $\delta((S_1s)^{R_1r}) \ = \ \delta(S_1s^r) \ = \ R_1\partial(s^r) \ = \ R_1r^{-1}(\partial s)r \ = \ (R_1r^{-1})(R_1\partial s)(R_1r) \ = \ (\delta(S_1s))^{R_1r}$,

M2: $(S_1s')^{\delta(S_1s)} \ = \ (S_1s')^{R_1(\partial s)} \ = \ S_1(s')^{\partial s} \ = \ S_1s^{-1}s's \ = \ (S_1s^{-1})(S_1s')(S_1s)$.

□

Theorem 1.8 (from Tim Porter's Topology paper [50])
Every crossed module is a quotient of normal inclusion crossed modules.

Proof:

$$\begin{array}{ccccc} 1 & \xrightarrow{\;\;0\;\;} & S & \xrightarrow{\;\;1\;\;} & S \\ {\scriptstyle 0}\big\downarrow & & \big\downarrow{\scriptstyle \epsilon} & & \big\downarrow{\scriptstyle \partial} \\ S & \xrightarrow[\;\;\zeta\;\;]{} & R \ltimes S & \xrightarrow[\;\;h\;\;]{} & R \\ & & & & \\ \Gamma_1\mathcal{X} & & \Gamma_0\mathcal{X} & & \mathcal{X} \end{array} \qquad (3)$$

Given $\mathcal{X} = (\partial : S \to R)$, the crossed modules $\Gamma_0\mathcal{X}$ and $\Gamma_1\mathcal{X}$ are given by

- $\epsilon s = (1, s), \quad \zeta s = (\partial s, s^{-1}), \quad {s_1}^{(r,s)} = s^{-1}{s_1}^r s = {s_1}^{r(\partial s)}$,
- $h(r, s) = r(\partial s), \quad (r_1, s)^r = (r^{-1}r_1 r, s^r)$.

Verification that $(\epsilon : S \to R \ltimes S)$ is a crossed module.

M1: $\left\{ \begin{array}{rcl} \epsilon(s_1^{(r,s)}) & = & \epsilon(s^{-1}s_1^r s) \;=\; (1, s^{-1}s_1^r s), \\ (r,s)^{-1}(\epsilon s_1)(r,s) & = & (r^{-1}, (s^{-1})^{r^{-1}})(1, s_1)(r, s) \;=\; (r^{-1}1r, (s^{-1})^{r^{-1}r}s_1^r s) \;=\; (1, s^{-1}s_1^r s). \end{array} \right.$

M2: ${s_1}^{\epsilon s} \;=\; {s_1}^{(1,s)} \;=\; s^{-1}s_1 s.$

Verification that $(1, h)$ and (ϵ, δ) are morphisms of crossed modules.

(i) $h\epsilon s \;=\; h(1, s) \;=\; \partial s$,

(ii) $\left\{ \begin{array}{rcl} 1({s_1}^{(r,s)}) & = & {s_1}^{(r,s)} \;=\; s^{-1}{s_1}^r s, \\ (1s_1)^{h(r,s)} & = & {s_1}^{r(\partial s)} \;=\; s^{-1}{s_1}^r s. \end{array} \right.$

(ii′) $\left\{ \begin{array}{rcl} \epsilon({s_1}^s) & = & (1, {s_1}^s) \;=\; (1, s^{-1}s_1 s), \\ (\epsilon s_1)^{\partial s} & = & (1, s_1)^{\partial s} \;=\; (1, s_1)^{(1,\partial s)} \;=\; (1, {s_1}^{\partial s}) \;=\; (1, s^{-1}s_1 s). \end{array} \right.$

Verification that $(0, \zeta)$ is the inclusion of a normal sub-crossed module.

We only need to show that ${s_1}^{-1}{s_1}^{\zeta s} = 1$.

$${s_1}^{-1}{s_1}^{\zeta s} \;=\; {s_1}^{-1}{s_1}^{(\partial s, s^{-1})} \;=\; {s_1}^{-1}s{s_1}^{\partial s}s^{-1} \;=\; 1.$$

Verification that $h\zeta = 0$.

$$h\zeta s \;=\; h(\partial s, s^{-1}) \;=\; (\partial s)(\partial s^{-1}) \;=\; 1.$$

□

Corollary 1.9 *In the previous result* $\Gamma_0, \Gamma_1 : \mathsf{XMod} \to \mathsf{XMod}$ *are functors and* $(0, \zeta) : \Gamma_1 \to \Gamma_0$ *is a natural transformation.*

Note that, in diagram (3), $(h : R \ltimes S \to R)$ is *not* in general a crossed module, so the right-hand square is not a crossed square.

We may extend the above Theorem to morphisms of crossed modules as follows.

[**Complicated diagram to be inserted here.**]

1.6 Peiffer subgroup of a pre-crossed module

We shall construct from any pre-crossed module a crossed module. Given $\mathcal{Q} = (\delta : Q \to R)$ the *Peiffer commutators* of Q are elements of the form

$$\langle q_1, q_2 \rangle \quad = \quad (q_2^{-1})^{q_1} \, {q_2}^{\delta q_1}, \qquad \text{where} \quad q_1, q_2 \in Q,$$

so that

$$\langle q_1, q_2 \rangle \; = \; 1 \quad \Leftrightarrow \quad {q_2}^{\delta q_1} \; = \; q_1^{-1} \, q_2 \, q_1$$

The subgroup P of Q generated by the Peiffer commutators is known as the *Peiffer subgroup* of $\mathcal{Q}$.

Lemma 1.10

(a) $\delta\langle q_1, q_2 \rangle \; = \; 1_R$;

(b) $\langle q_1 q_2, q_3 \rangle \; = \; \langle q_1, q_3 \rangle^{q_2} \, \langle q_2, {q_3}^{\delta q_1} \rangle$;

(c) $\langle q_1, q_2 q_3 \rangle \; = \; \langle q_1, q_3 \rangle \, \langle q_1, q_2 \rangle^{{q_3}^{\delta q_1}}$;

(d) $\langle q_1, q_2 \rangle^r \; = \; \langle {q_1}^r, {q_2}^r \rangle$.

Proof: The first part follows from $\delta({q_2}^{\delta q_1}) \; = \; (\delta q_1)^{-1}(\delta q_2)(\delta q_1)$. The remaining parts follow on expanding the Peiffer elements, and are closely related to identities for commutators and crossed pairings (see Subsection 4.2):

$$\begin{aligned}
\langle q_1 q_3, q_2 \rangle &= q_3^{-1} q_1^{-1} q_2^{-1} q_1 \{ {q_2}^{\delta q_1} q_3 q_3^{-1} ({q_2}^{\delta q_1})^{-1} \} q_3 {q_2}^{\delta(q_1 q_3)} \; = \; \langle q_1, q_2 \rangle^{q_3} \, \langle q_3, {q_2}^{\delta q_1} \rangle \, , \\
\langle q_1, q_2 q_3 \rangle &= q_1^{-1} q_3^{-1} \{ q_1 {q_3}^{\delta q_1} ({q_3}^{\delta q_1})^{-1} q_1^{-1} \} q_2^{-1} q_1 (q_2 q_3)^{\delta q_1} \; = \; \langle q_1, q_3 \rangle \, \langle q_1, q_2 \rangle^{{q_3}^{\delta q_1}} \, , \\
\langle {q_1}^r, {q_2}^r \rangle &= (q_1^{-1})^r (q_2^{-1})^r {q_1}^r ({q_2}^r)^{(r^{-1}(\delta q_1) r)} \; = \; (q_1^{-1} q_2^{-1} q_1 {q_2}^{\delta q_1})^r \, .
\end{aligned}$$

□

The following result follows immediately from these identities.

Corollary 1.11 *Let $\mathcal{Q} = (\delta : Q \to R)$ be a pre-crossed module. Then the set P of Peiffer commutators in $\mathcal{Q}$ is a subgroup of* $\ker \delta$*; is normal in Q; and is R-invariant.*

Proposition 1.12 *If $(\ddot{\alpha}, \dot{\alpha}) : \mathcal{Q}_1 \to \mathcal{Q}_2$ is a morphism of pre-crossed modules, then Peiffer commutators in Q_1 are mapped to Peiffer commutators in Q_2.*

Proof: The source map $\ddot{\alpha}$ is compatible with the Peiffer pairing:

$$\begin{aligned}
\ddot{\alpha}\langle q_1, q_2 \rangle &= (\ddot{\alpha} q_1)^{-1} \, (\ddot{\alpha} q_2)^{-1} \, (\ddot{\alpha} q_1) \, (\ddot{\alpha}({q_2}^{\delta_1 q_1})) \\
&= (\ddot{\alpha} q_1)^{-1} \, (\ddot{\alpha} q_2)^{-1} \, (\ddot{\alpha} q_1) \, (\ddot{\alpha} q_2)^{\dot{\alpha}\delta_1 q_1} \\
&= (\ddot{\alpha} q_1)^{-1} \, (\ddot{\alpha} q_2)^{-1} \, (\ddot{\alpha} q_1) \, (\ddot{\alpha} q_2)^{\delta_2(\ddot{\alpha} q_1)} \\
&= \langle \, \ddot{\alpha} q_1, \, \ddot{\alpha} q_2 \, \rangle \, .
\end{aligned}$$

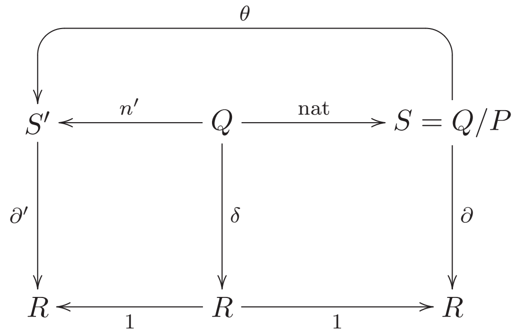
□

Proposition 1.13 *Let $\mathcal{Q} = (\delta : Q \to R)$ be a pre-crossed module. Then there is a crossed module $\mathcal{X} = (\partial : S \to R)$ and a morphism of pre-crossed modules* $(\mathrm{nat}, 1) : \mathcal{Q} \to \mathcal{X}$ *such that* $(\mathrm{nat}, 1)$ *is universal for morphisms from $\mathcal{Q}$ to crossed modules over R.*

Proof: Let P be the Peiffer group of $\mathcal{Q}$. Then the quotient group $S = Q/P$ is well-defined, $\text{nat} : Q \to S$ is the natural quotient map, and S inherits an R-action and an R-morphism:

$$(Pq)^r = P(q^r) \ , \qquad\qquad \partial : S \to R,\ (Pq) \mapsto \delta q \ .$$

So $\mathcal{X} = (\partial : S \to R)$ is a pre-crossed module. By definition of P, we have $s_1^{-1}s_2s_1 = {s_2}^{\partial s_1}$ for all $s_1, s_2 \in S$, and so $\mathcal{X}$ is a crossed module. The quotient morphism $(\text{nat}, 1)$ is clearly a morphism of pre-crossed modules.



If $\mathcal{X}' = (\partial' : S' \to R)$ is a crossed module and if $(n', 1) : \mathcal{Q} \to \mathcal{X}'$ is a pre-crossed module morphism, then there is a unique crossed module morphism $(\theta, 1) : \mathcal{X} \to \mathcal{X}'$ such that $(\theta, 1) \circ (\text{nat}, 1) = (n', 1)$ where $\theta(Pq) = n'q$. □

The following result, which gives a normal generating set for the Peiffer group, is Proposition 3 of [14]. The method of proof was suggested by Philip Higgins.

Proposition 1.14 *Given the following ingredients:*

- $\mathcal{Q} = (\delta : Q \to R)$, *a pre-crossed module;*
- Γ, *a generating set for Q, closed under the action of R;*
- P, *the Peiffer group of $\mathcal{Q}$;*
- E, *the set of Peiffer elements $\{\langle a, b\rangle \mid a, b \in \Gamma\}$;*

then P is the normal closure of E in Γ.

Proof: Let P' be the normal closure of E in Q, so that

$$P' \ \trianglelefteq\ P \ \trianglelefteq\ \ker \delta \ \trianglelefteq\ Q \ .$$

If $P \neq P'$ then there is some $z = \langle x, y\rangle \in P \setminus P'$ such that $P'z \neq P'$ in Q/P' . Since

$$P'z \neq P' \quad \Rightarrow \quad zP' \neq P' \quad \Rightarrow \quad y^{\delta x}\, P' \neq (x^{-1}yx)P' \quad \Rightarrow \quad P'y^{\delta x} \neq P'(x^{-1}yx) \ ,$$

we need to show that $P'\, y^{\delta x} \ = \ P'(x^{-1}yx)$ for all $\langle x, y\rangle \in P'$.

Since Γ is R-invariant, part (c) of Lemma 1.10 shows that E is R-invariant, and hence P' is R-invariant. Since $P' \trianglelefteq Q$, form $S' = Q/P'$ with R-action $(P'q)^r = P'(q^r)$. The homomorphism δ induces $\partial' : S' \to P'$, $P'q \mapsto \delta q$, and $\mathcal{X}' = (\partial' : S' \to R)$ is a pre-crossed module since

$$\partial'((P'q)^r) \ = \ \partial'(P'(q^r)) \ = \ \delta q^r \ = \ r^{-1}(\delta q)r \ = \ r^{-1}(\partial'(P'q))r \ .$$

Since Γ generates Q as a group, the set of cosets $C' = \{P'a \mid a \in \Gamma\}$ generates S' and, for all $P'a, P'b \in C'$,

$$(P'a)^{\partial'(P'b)} = (P'a)^{\delta b} = P'(a^{\delta b}) = P'(b^{-1}ab) = (P'b)^{-1}(P'a)(P'b) . \tag{4}$$

For fixed b, the set of $P'a$ satisfying (4) is a subgroup of S' :

$$((P'a)(P'c))^{\partial'(P'b)} = (P'a)^{\partial'(P'b)}(P'c)^{\partial'(P'b)} = (P'b)^{-1}(P'a)(P'c)(P'b) .$$

So (4) is true for all $(P'a) \in S'$ and all $(P'b) \in C'$.

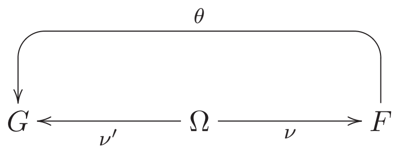
Also, the set of $(P'b) \in C'$ satisfying (4) is closed under multiplication and inversion since

- $(P'a)^{\partial'(P'(bc))} = ((P'a)^{\delta(b)})^{\delta c} = (P'(bc))^{-1}\,(P'a)\,(P'(bc))$;
- $(P'a)^{\partial'(P'b)^{-1}} = P'c \quad \Rightarrow \quad P'a = (P'b)^{-1}(P'c)(P'b) \quad \Rightarrow \quad (P'b))(P'a)(P'b)^{-1} = P'c$.

Thus (4) holds for *all* $P'a, P'b \in C'$. □

1.7 Free Crossed Modules

We first recall a property of free groups which we wish to generalise. Let Ω be a set. The *free group on* Ω is a group F and a function $\nu : \Omega \to F$ such that if G is a group and $\nu' : \Omega \to G$ a function, then there exists a unique group homomorphism $\theta : F \to G$ such that $\theta \circ \nu = \nu'$.



To construct a particular model for $F = F(\Omega)$ we take an alphabet consisting of all the elements of Ω together with their formal inverses, and take for F the set of all reduced words in this alphabet with concatenation as the group product and the empty word as the identity. The details of this construction should be familiar to the reader.

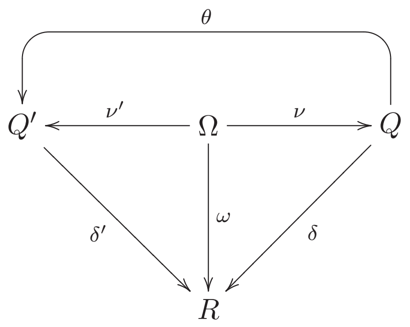
We now define a *free pre-crossed module* in an analogous manner. The ingredients for the construction are

- a set Ω,
- a group R,
- a function $\omega : \Omega \to R$.

The resulting construction consists of

- a pre-crossed module $\mathcal{Q} = (\delta : Q \to R)$,
- a function $\nu : \Omega \to Q$ such that $\delta \circ \nu = \omega$.

The universal property required of this construction is that if $\mathcal{Q}' = (\delta' : Q' \to R)$ is another pre-crossed module, and if $\nu' : \Omega \to Q'$ satisfies $\delta' \circ \nu' = \omega$, then there exists a unique morphism of pre-crossed modules $(\theta, 1) : \mathcal{Q} \to \mathcal{Q}'$ such that $(\theta, 1) \circ (\nu, 1) = (\nu', 1)$.



A particular model is obtained as follows:

- the source group Q is the free group $F(\Omega \times R)$,
- the boundary map is defined on generators by $\delta(\rho, r) = (\omega\rho)^r = r^{-1}(\omega\rho)r$,
- the action is given by $(\rho, r)^{r'} = (\rho, rr')$,
- the function is given by $\nu(\rho) = (\rho, 1)$.

We observe that

$$\delta \circ \nu(\rho) \;=\; \delta(\rho, 1) \;=\; \omega\rho \quad \text{for all} \quad \rho \in \Omega,$$

and verify **X1:** as follows:

$$\delta((\rho, r)^{r'}) \;=\; \delta(\rho, rr') \;=\; r'^{-1}(r^{-1}(\omega\rho)r)r' \;=\; r'^{-1}\,\delta(\rho, r)\,r' \;.$$

To check the universal property we need to define $\theta : Q \to Q'$ in such a way that $\theta(\rho, 1) = \nu'\rho$. Since θ is to preserve the R-action, we are forced to define

$$\theta(\rho, r) \;=\; \theta((\rho, 1)^r) \;=\; (\nu'\rho)^r \;.$$

This defines θ on the whole of Q, and we verify that $(\theta, 1)$ is a morphism of pre-crossed modules:

- $\delta'\theta(\rho, r) \;=\; \delta'((\nu'\rho)^r) \;=\; r^{-1}(\delta'\nu'\rho)r \;=\; r^{-1}(\omega\rho)r \;=\; \delta(\rho, r)\;,$
- $\theta((\rho, r)^{r'}) \;=\; \theta(\rho, rr') \;=\; (\nu'\rho)^{rr'} \;=\; (\theta(\rho, r))^{r'}\;.$

We are now in a position to construct the free crossed module associated to a free pre-crossed module, simply by factoring out the Peiffer commutators in $Q = F(\Omega \times R)$:

$$\begin{aligned}
\langle(\rho_1, r_1), (\rho_2, r_2)\rangle \;&=\; (\rho_1, r_1)^{-1}\,(\rho_2, r_2)^{-1}\,(\rho_1, r_1)\,(\rho_2, r_2)^{\delta(\rho_1, r_1)} \\
&=\; (\rho_1, r_1)^{-1}\,(\rho_2, r_2)^{-1}\,(\rho_1, r_1)\,(\rho_2, r_2)^{r_1^{-1}(\omega\rho_1)r_1} \\
&=\; (\rho_1, r_1)^{-1}\,(\rho_2, r_2)^{-1}\,(\rho_1, r_1)\,(\rho_2, r_2 r_1^{-1}(\omega\rho_1)r_1)\;.
\end{aligned}$$

We thus obtain a crossed module with source $C(\omega) = Q/P$:

$$\begin{aligned}
\mathcal{F}_\rho \;&=\; (\partial : C(\omega) \to R)\,, \\
\partial(P(\rho, r)) \;&=\; \delta(\rho, r) \;=\; r^{-1}(\omega\rho)r\,, \\
(P(\rho, r))^{r'} \;&=\; P(\rho, rr')\,.
\end{aligned}$$

It is often convenient to use an alternative notation for elements in Q. Since $(\rho, r) = (\rho, 1)^r$ we may drop the "$, 1$" and write $(\rho)^r$ for (ρ, r), and the inverse element by $(\rho^{-1})^r$:

$$\{(\rho)^r\}^{-1} \ = \ (\rho, r)^{-1} \ = \ \{(\rho, 1)^r\}^{-1} \ = \ \{(\rho, 1)^{-1})^r \ = \ (\rho^{-1})^r \ .$$

The action is then given by

$$((\rho^\epsilon)^r)^{r'} \ = \ (\rho^\epsilon)^{(rr')} \quad \text{where} \quad \epsilon = \pm 1 \ .$$

The Peiffer commutators on generators in this notation are

$$\langle\, ({\rho_1}^{\epsilon_1})^{r_1},\, ({\rho_2}^{\epsilon_2})^{r_2}\,\rangle \ = \ ({\rho_1}^{-\epsilon_1})^{r_1}\, ({\rho_2}^{-\epsilon_2})^{r_2}\, ({\rho_1}^{\epsilon_1})^{r_1}\, ({\rho_2}^{\epsilon_2})^{r_2 r_1^{-1}(\omega\rho_1) r_1} \ .$$

1.8 The monoid version of free crossed modules

Given $Y = \Omega \times R$, define

$$\bar{Y} \ = \ \{y^+ : y \in Y\} \sqcup \{y^- : y \in Y\}\, .$$

It is convenient to use an alternative notation, as above:

$$(\rho^+)^r \ := \ (\rho, r)^+\, , \qquad (\rho^-)^r \ := \ (\rho, r)^-\, .$$

Then $H = \bar{Y}^*$ is the free monoid on $\bar{Y}$ with empty word λ and elements

$$(\rho_1^{\epsilon_1})^{r_1}\, (\rho_2^{\epsilon_2})^{r_2} \cdots (\rho_n^{\epsilon_n})^{r_n}\, , \quad \rho_i \in \Omega,\ \epsilon_i \in \{+, -\},\ r_i \in R\, .$$

The boundary map is the monoid morphism

$$\bar{\delta}\ :\ \bar{Y}^* \to R, \quad (\rho^+)^r \mapsto r^{-1}(\omega\rho)r, \quad (\rho^-)^r \mapsto r^{-1}(\omega\rho)^{-1}r.$$

Then $F(\omega)$ is the quotient monoid $\bar{Y}^*/\equiv$ where $\equiv$ is the congruence generated by

- inverse pairs $(y^\epsilon y^{-\epsilon}, \lambda)$,
- Peiffer pairs $(y^{-\epsilon} z^\eta y^\epsilon, (z^\eta)^{\bar{\delta} y^\epsilon})$, $\quad \epsilon, \eta \in \{+, -\}, \quad -(-) = +, \quad$ etc.

In the special case of a group presentation

$$\mathcal{P} \ = \ \mathrm{grp}(X, \omega : \Omega \to F(X)),$$

$\omega\rho$ is a relator and so a word in $F(X)$. Then $Y = \Omega \times F(X)$ and $H = \bar{Y}^*$ has elements of the form

$$(\rho_1^{\epsilon_1})^{u_1}\, (\rho_2^{\epsilon_2})^{u_2} \cdots (\rho_n^{\epsilon_n})^{u_n}\, ,$$

and $\ker \partial = \Pi_2(\mathcal{P})$ is the $\mathbb{Z}G$-module of identities among the relators of $\mathcal{P}$.

Example 1.15 Consider the following presentation for the quaternion group of size 8.

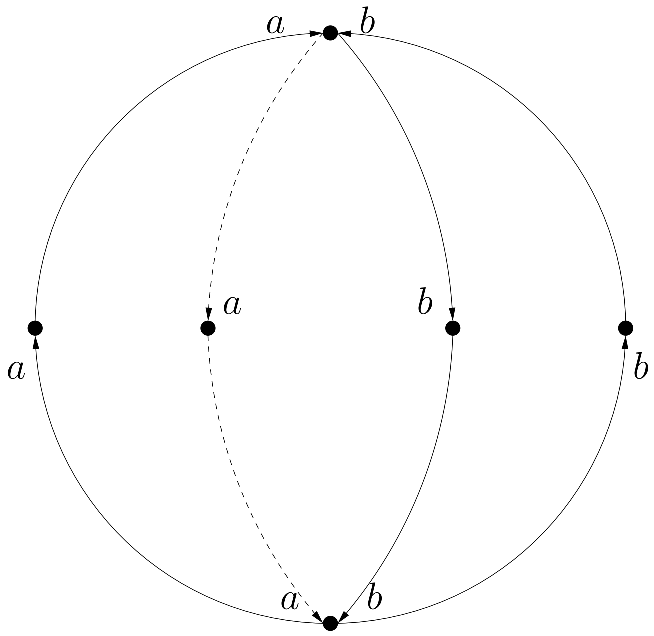
$$X = \{a, b\}, \quad \Omega = \{\rho_1, \rho_2, \rho_3, \rho_4\}, \quad \omega : \rho_1 \mapsto a^4,\ \rho_2 \mapsto b^4,\ \rho_3 \mapsto abab^{-1},\ \rho_4 \mapsto a^2b^2.$$

The identity

$$\iota \ = \ (\rho_4^-)\ (\rho_1^+)^{a^2}\ (\rho_4^-)^{a^2}\ (\rho_2^+)$$

maps by ∂ to

$$(b^{-2}a^{-2}).a^{-2}(a^4)a^2.a^{-2}(b^{-2}a^{-2})a^2.(b^4) \ = \ \lambda\,.$$



In the van Kampen diagrams, the four relators tile a sphere as shown above. Tracing out $\partial\iota$ we walk around the boundaries, with every edge cancelling out with its inverse.

1.9 A Geometric Example of Crossed Modules

The major geometric example of a crossed module can be expressed in two ways. Let (X, A, a) be a based pair of spaces, with $a \in A \subseteq X$. The *second relative homotopy group* $\pi_2(X, A, a)$ consists of homotopy classes rel J^1 of continuous maps

$$\alpha \;:\; (I^2, \dot{I}^2, J^1) \;\to\; (X, A, a)$$

where $I = [0, 1]$ and $J^1 = (\{0, 1\} \times I) \cup (I \times \{1\}) \subset I^2$. Each such α is a map from the unit square I^2 to the space X mapping the left, top, and right sides of the square to the point a and the bottom side to a loop β_α at a. We may represent such a map by the diagram in Figure 1.

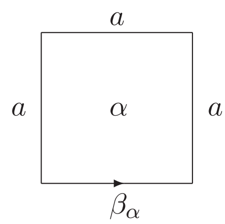


Figure 1: An element $\alpha \in \pi_2(X, A, a)$

Recall that the fundamental group $\pi_1(A, a)$ consists of maps $\gamma : I \to A,\ \gamma(0) = \gamma(1) = a$. With such a γ it is easy to construct maps $I^2 \to A$ which map two sides of the square to a and two sides to γ:

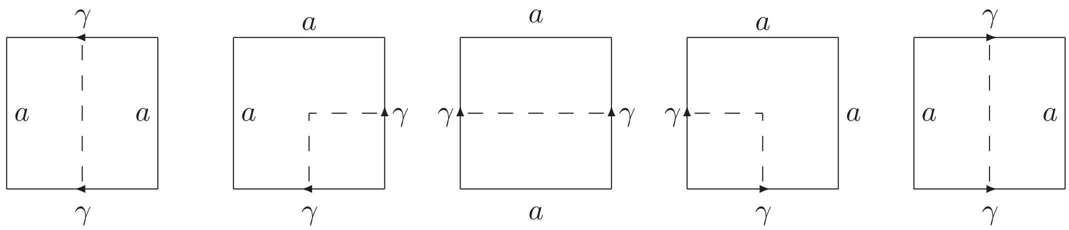


Figure 2: Five maps $I^2 \to A$ derived from $\gamma \in \pi_1(A, a)$

Whitehead showed in [53] that there is a crossed module $\Pi_2(X, A, a)$ with boundary map

$$\partial \;:\; \pi_2(X, A, a) \to \pi_1(A, a), \quad \alpha \mapsto \beta_\alpha = \alpha(I \times \{0\}) \;.$$

The image of $\alpha \in \pi_2(X, A, a)$ under the action of $\gamma \in \pi_1(A, a)$ is illustrated in Figure 3, surounding α with the five maps in Figure 2. Note that the boundary loop is the conjugate $\gamma^{-1}\beta_\alpha\gamma$.

The meaning of this composite square is as follows. Squares may be joined along an edge when the values agree on that edge. If a composite is then p units across by q units high, scaling factors $1/p$ horizontally and $1/q$ vertically are used to obtain a new map from I^2 to X.

Figure 4 gives an outline verification of the second crossed module axiom for $\Pi_2(X, A, a)$, where a square marked a represents the constant map $I^2 \to \{a\}$.

Whitehead's main result in [52, 53, 55] was:

Theorem 1.16 (Whitehead) *If X is obtained from A by attaching 2-cells, then $\pi_2(X, A, a)$ is isomorphic to the free crossed $\pi_1(A, a)$-module on the attaching maps of the 2-cells.*

[More here?]

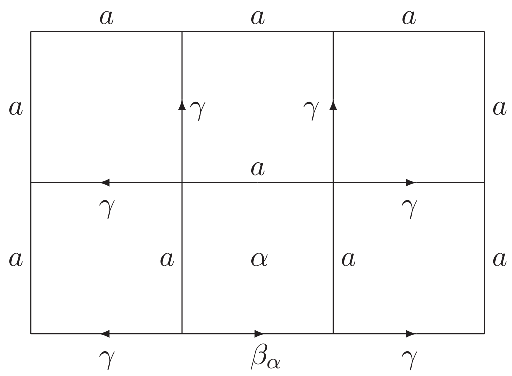


Figure 3: Action of γ on α

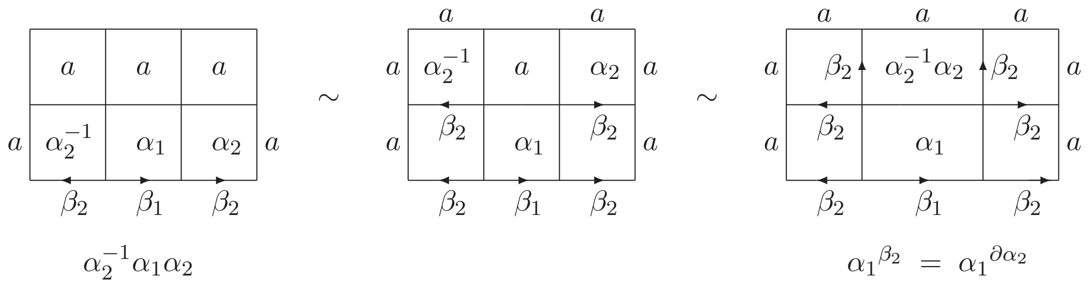


Figure 4: Verification of **X2:** for $\Pi_2(X, A, a)$.

1.10 Semidirect Products

We include here some basic results on semidirect products which will be needed in later sections.

Proposition 1.17

(a) *If a set X has a right G-action $x \mapsto x^g$ then X has an associated left G-action:*

$$ {}^g x \quad := \quad x^{g^{-1}} \ . $$

(b) *The semidirect products $R \ltimes S$ and $S \rtimes R$ have multiplication rules*

$$ \begin{aligned} (r,s)(q,t) &= (rq,\, s^q t) \quad \textit{in} \quad R \ltimes S \ , \\ (s,r)(t,q) &= (s\, {}^r t,\, rq) \quad \textit{in} \quad S \rtimes R \ . \end{aligned} $$

(c) *There is an isomorphism between these two groups:*

$$ \psi \ : \ R \ltimes S \to S \rtimes R \,, \quad (r,s) \mapsto ({}^r s,\, r) \ , $$

with inverse

$$ \psi^{-1} \ : \ S \rtimes R \to R \ltimes S \,, \quad (s,r) \mapsto (r,\, s^r) \ . $$

1.11 Cat1-groups

In [41] Loday reformulated the notion of a crossed module as a cat^1-group, namely a group G with a pair of endomorphisms $t, h : G \to G$ having a common image R and satisfying certain axioms. Alternatively we may define a cat^1-group $\mathcal{C} = (e; t, h : G \to R)$ to have source group G, range group R, and three homomorphisms: two surjections $t, h : G \to R$ and an embedding $e : R \to G$ as shown in the following diagram:

$$G \underset{e}{\overset{t,h}{\rightleftarrows}} R$$

These homomorphisms are required to satisfy the following axioms:

$$\begin{array}{ll} \textbf{C1:} & te(r) = he(r) = r \text{ for all } r \in R, \\ \textbf{C2:} & [\ker t, \ker h] = \{1_G\}. \end{array}$$

The maps t, h are usually referred to as the *source* and *target*, but we choose to call them the *tail* and *head* of $\mathcal{C}$, because *source* is the GAP term for the domain of a function.

A morphism $\mathcal{C}_1 \to \mathcal{C}_2$ of cat^1-groups is a pair (γ, ρ) where $\gamma : G_1 \to G_2$ and $\rho : R_1 \to R_2$ are homomorphisms satisfying

$$t_2\gamma = \rho t_1, \quad h_2\gamma = \rho h_1, \quad e_2\rho = \gamma e_1. \tag{5}$$

An arbitrary cat^1-group $\mathcal{C} = (e; t, h : G \to R)$ is isomorphic to the cat^1-group $\mathcal{C}' = (e'; t', h' : R \ltimes S \to R)$ where $S = \ker t$, the action of R on S is given by

$$s^r = s^{er} = (er)^{-1}s(er),$$

and the semidirect product $R \ltimes S$ has composition and inverse given by

$$(r_1, s_1)(r_2, s_2) \;=\; (r_1r_2, {s_1}^{r_2}s_2), \;\; (r, s)^{-1} \;=\; (r^{-1}, (s^{-1})^{r^{-1}}).$$

Since R acts on both itself (by conjugation) and on S, it also acts on G:

$$g^r \;=\; (er^{-1})g(er), \qquad (r_0, s_0)^r \;=\; (r^{-1}r_0r, s_0^r)\,. \tag{6}$$

The homomorphisms in $\mathcal{C}'$ are given by

$$t'(r, s) = r, \quad h'(r, s) = r(\partial s), \quad e'r = (r, 1). \tag{7}$$

Definition 1.18 *The* semidirect factorisation *of* $\mathcal{C}$ *is* $(\phi,\ \mathrm{id}_R) : \mathcal{C} \to \mathcal{C}'$, *where*

$$\phi \;:\; G \to R \ltimes S, \quad g \mapsto (tg, ug), \qquad \textit{where} \;\; ug = (etg^{-1})g. \tag{8}$$

The inverse isomorphism is $(\phi^{-1},\ \mathrm{id}_R) : \mathcal{C}' \to \mathcal{C}$, *where*

$$\phi^{-1} \;:\; R \ltimes S \to G, \quad (r, s) \mapsto (er)s.$$

Lemma 1.19 *The mapping* $u : G \to \ker t,\ g \mapsto (etg^{-1})g$ *has the following properties.*

(i) $u^2 = u$,

(ii) $tug = 1_R, \quad hug = (tg^{-1})(hg), \quad uer = 1_G$,

(iii) $u(g_1g_2) = (ug_2)(ug_1)^{g_2}$,

(iv) $(ug)^{-1} = g^{-1}(etg) = (u(g^{-1}))^g$.

The crossed module $\mathcal{X} = (\partial : S \to R)$ associated to $\mathcal{C}$ and $\mathcal{C}'$ has boundary $\partial = h|_S$ and action $s^r := s^{er}$. The cat^1-group $\mathcal{C} = \mathcal{C}'$ associated to $\mathcal{X} = (\partial : S \to R)$ has $G = R \ltimes S$, where the action is that in $\mathcal{X}$, and homomorphisms given by (7). We denote by ϵ the inclusion of S in G, so that $\partial = h\epsilon$.

Given a morphism $(\sigma, \rho) : \mathcal{X}_1 \to \mathcal{X}_2$ of crossed modules, the associated morphism of cat1-groups is $(\gamma, \rho) : \mathcal{C}_1 \to \mathcal{C}_2$ where $\gamma(r_1, s_1) = (\rho r_1, \sigma s_1)$. Similarly, given a morphism $(\gamma, \rho) : \mathcal{C}_1 \to \mathcal{C}_2$ of cat1-groups, the associated morphism of crossed modules is $(\sigma, \rho) : \mathcal{X}_1 \to \mathcal{X}_2$ where $\sigma s = \gamma(1, s)$.

George Janelidze has noted the following variant of the second cat^1-group axiom:

$$\textbf{C2}'\textbf{:} \qquad [ug_1, ug_2] \ = \ 1_G \qquad \text{for all } \ g_1, g_2 \in G \ .$$

It follows that cat^1-groups form an equational variety.

Lemma 1.20 *If $\mathcal{C} = (e; t, h : G \to R)$ is a cat^1-group then there is a group homomorphism*

$$(t, h) \ : \ G \to R \times R, \ \ g \mapsto (tg, hg) \ .$$

Proposition 1.21 (Comment by Tim Porter during a seminar on 18/10/02.)
A congruence $\equiv$ *(in the sense of congruence on a monoid) on a group R gives rise to a cat^1-group.*

Proof: The set of equivalent pairs,

$$G \ = \ \{(r_1, r_2) \in R \times R \ \mid \ r_1 \equiv r_2\}$$

is a subgroup of $R \times R$. The required $\mathcal{C} = (e; t, h : G \to R)$ has homomorphisms given by:

$$t(r_1, r_2) = r_1, \qquad h(r_1, r_2) = r_2, \qquad e(r) = (r, r).$$

The associated crossed module has source

$$\ker t \ = \ \{(r, 1) \mid r \equiv 1\}$$

which shows that the elements equivalent to 1_R in the congruence form a normal subgroup. □

1.12 Pre-cat^1-groups

When axioms **X2:** and **C2:** are *not* satisfied by $\mathcal{Q} = (\delta : Q \to R)$ and $\mathcal{B} = (e; t, h : R \ltimes Q \to R)$, the corresponding structures are known as *pre-crossed modules* and *pre-cat^1-groups*. In this case recall from Subsection 1.6 that the *Peiffer subgroup* P of Q is the subgroup of $\ker(\delta)$ generated by *Peiffer commutators*

$$\langle q_1, q_2 \rangle \ = \ q_1^{-1} \, q_2^{-1} \, q_1 \, {q_2}^{\partial q_1} \ .$$

Then $\mathcal{P} = (0 : P \to \{1_R\})$ is a normal sub-pre-crossed module of $\mathcal{Q}$ and $\mathcal{X} = \mathcal{Q}/\mathcal{P} = (\partial : S = Q/P \to R)$ is a crossed module. The restriction of $\epsilon : Q \to R \ltimes Q$ to P is given by

$$\epsilon\langle q_1, q_2 \rangle = [(\delta q_1^{-1}, q_1), (1_R, {q_2}^{\delta q_1})] \in [\ker h, \ker t].$$

The image ϵP is the Peiffer subgroup $[\ker h, \ker t]$ of $R \ltimes Q$ and, if ι is the inclusion $\{1_R\} \to R$, then $\mathcal{C}/(\epsilon, \iota)\mathcal{P} = (e; t, h : (R \ltimes Q)/\epsilon P \to R)$ is the cat^1-group corresponding to $\mathcal{X} = \mathcal{Q}/\mathcal{P}$.

1.13 Group Groupoids

Cat1-groups may also be thought of as group-groupoids. A group groupoid is a set which has both a group structure and a groupoid structure (see subsection **??**). From a categorical viewpoint, it is both a group object in the category of groupoids and a groupoid object in the category of groups.

The underlying groupoid $\mathcal{G}$ of a cat^1-group $\mathcal{C}$ has the group R as object set G_0 and the group G as the set of arrows G_1. The identity arrow at r is $1_r = er$. For each arrow g the tail (source) is tg and the head (target) is hg. Arrows g_1, g_2 are composable only when $hg_1 = tg_2 = r_1$ (say), in which case the composite arrow is

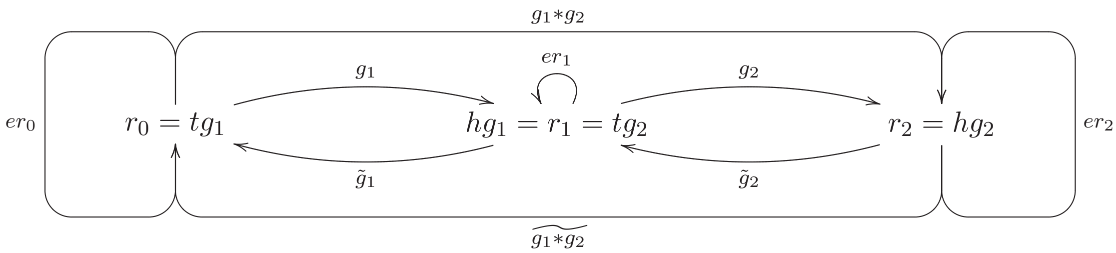
$$g_1 * g_2 \;=\; g_1(er_1{}^{-1})g_2 \quad \text{where} \quad t(g_1 * g_2) = tg_1 = r_0, \quad h(g_1 * g_2) = hg_2 = r_2. \tag{9}$$

This composition is, of course, associative:

$$g_1 * g_2 * g_3 \;=\; g_1(er_1{}^{-1})g_2(er_2{}^{-1})g_3.$$

The groupoid inverse $\tilde{g}$ of g for the composition is given by

$$\tilde{g} \;=\; (ehg)g^{-1}(etg) \quad \text{with} \quad t\tilde{g} = hg,\; h\tilde{g} = tg,\; g * \tilde{g} = etg \;\text{ and }\; \tilde{g} * g = ehg\ .$$



Hence the composites of one element with the groupoid inverse of another, when defined, are given by

$$\tilde{g}_1 * g_3 \;=\; (ehg_1)g_1^{-1}g_3 \qquad \text{and} \qquad g_4 * \tilde{g}_2 \;=\; g_4g_2^{-1}(etg_2). \tag{10}$$

The equivalent formulae for composition and inverse when $R \ltimes S$ replaces G are:

$$(r,s) * (r(\partial s), s') = (r, ss') \quad \text{and} \quad \widetilde{(r,s)} = (r(\partial s), s^{-1})\ .$$

Since $g^{-1}(etg) \in \ker t$ and $(ehg)g^{-1} \in \ker h$, the map $g \mapsto \tilde{g}$ is an automorphism of $\mathcal{G}$ which restricts to the identity map on eR and provides a cat^1-isomorphism from $\mathcal{C}$ to the *reverse* cat^1-group $\tilde{\mathcal{C}} = (e; h, t : G \to R)$ of $\mathcal{C}$. The set of arrows *out* from 1_R is $\ker t$ while the set of arrows *in* to 1_R is $\ker h$, so $\ker \partial$ is the set of loops at 1_R. The set of objects in the component of $\mathcal{G}$ connected to 1_R is the image of ∂, so $\mathcal{G}$ is discrete when $\partial = 0$.

Alternatively, starting with a group groupoid $\mathcal{G} = (G, t, h)$, define

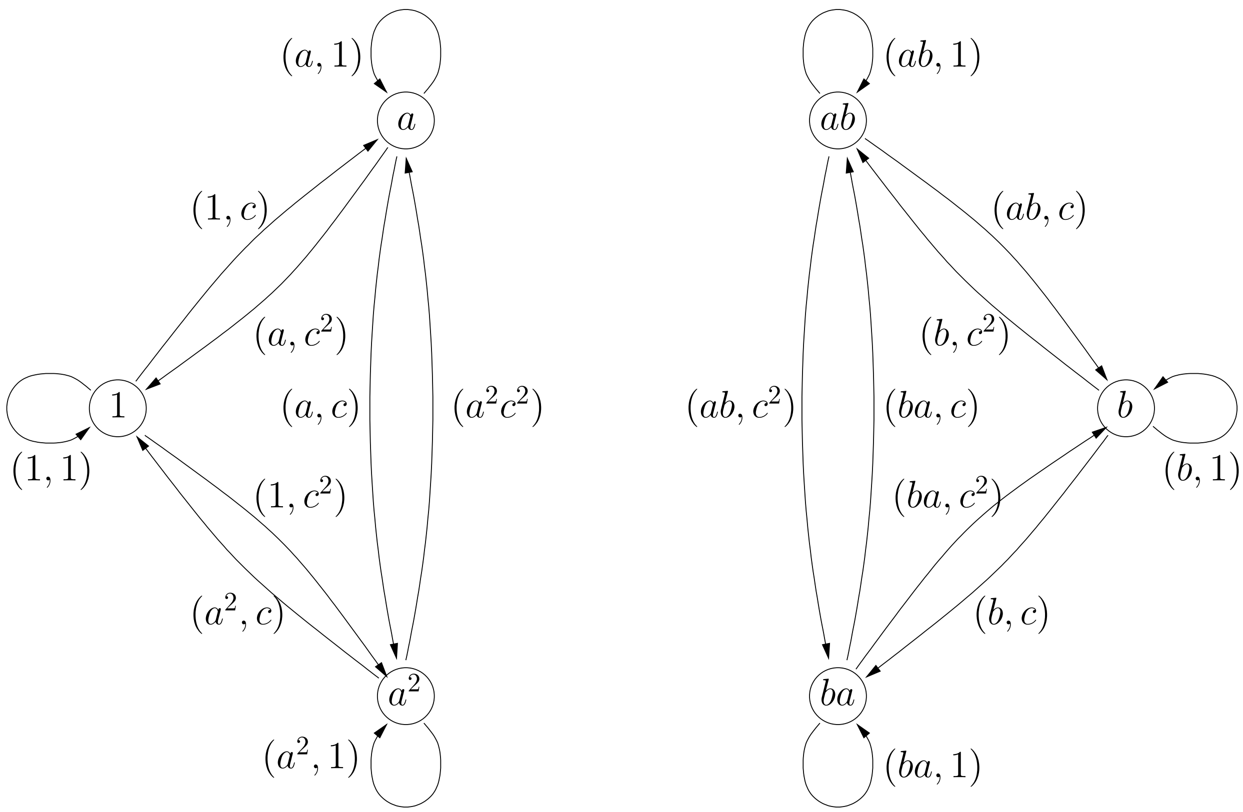
$$\begin{array}{rcl} R &=& \operatorname{im} t \;=\; \operatorname{im} h, \\ S &=& \{g \mid tg = 1\} \;=\; \ker t \;=\; \text{arrows out from } 1_R, \\ s^r &=& (er)^{-1}s(er), \quad \text{where } er \text{ is the identity loop at } r. \end{array}$$

See Subsection 2.3 for the group-groupoid equivalent of derivations and sections.

Example 1.22 The injection crossed module $X_3 = (1 : C_3 \to S_3)$ of the cyclic group $C_3 = \langle c \mid c^3 \rangle$ in the symmetric group $S_3 = \langle a, b \mid a^3, b^2, (ab)^2 \rangle$, with conjugation action $c^a = c, c^b = c^2$, has associated cat^1-group $(e; t, h : S_3 \ltimes C_3 \to S_3)$. The images of the tail and head functions are given in the following table:

g	tg	hg	g	tg	hg
$(1,1)$	1	1	$(b,1)$	b	b
$(1,c)$	1	a	(b,c)	b	ba
$(1,c^2)$	1	a^2	(b,c^2)	b	ab
$(a,1)$	a	a	$(ab,1)$	ab	ab
(a,c)	a	a^2	(ab,c)	ab	b
(a,c^2)	a	1	(ab,c^2)	ab	ba
$(a^2,1)$	a^2	a^2	$(ba,1)$	ba	ba
(a^2,c)	a^2	1	(ba,c)	ba	ab
(a^2,c^2)	a^2	a	(ba,c^2)	ba	b

The corresponding group-groupoid has 6 objects, 18 morphisms, 2 connected components, and may be pictured as:



We may compare the group multiplication with the groupoid multiplication by calculating, for example,

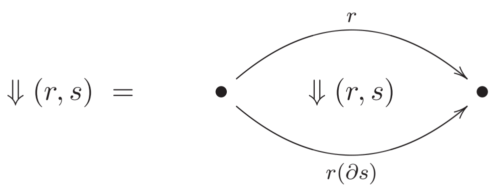
$$\begin{aligned} (a,c)(a^2,c) &= (1, c^{a^2}c) = (1,c^2), \\ (a,c) * (a^2,c) &= (a,c)(a^2,1)^{-1}(a^2,c) = (a^4, c^{a^3}c) = (a,c^2). \end{aligned}$$

1.14 2-groups

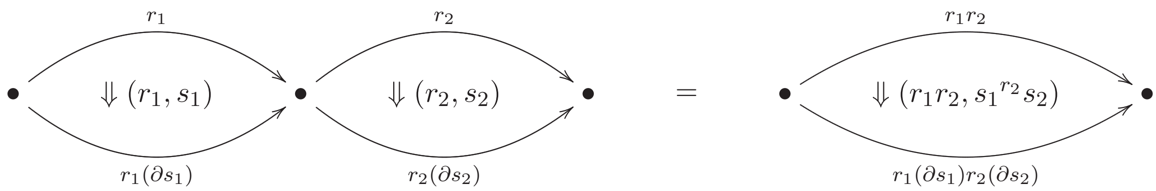
Finally, we think of such a structure as a special case of a 2-category, which has objects, morphisms, and 2-*cells*. We follow the presentation in Subsection 1.2.3 of Forrester-Barker's thesis [32]. For an introduction to 2-groupoids, see Kamps and Porter [40]. Note that a 2-group is *not* a special case of the group theorist's p-group, with $p = 2$, but is a 2-category with one object having all morphisms and 2-cells invertible.

The 2-group $\mathcal{H}$ associated to $\mathcal{X} = (\partial : S \to R)$ has

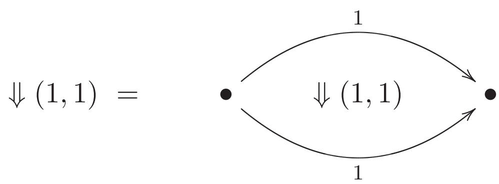
- a single object $\bullet$,
- morphisms $r \in R$,
- 2-cells $(r, s) \in R \ltimes S$ with tail r and head $r(\partial s)$.



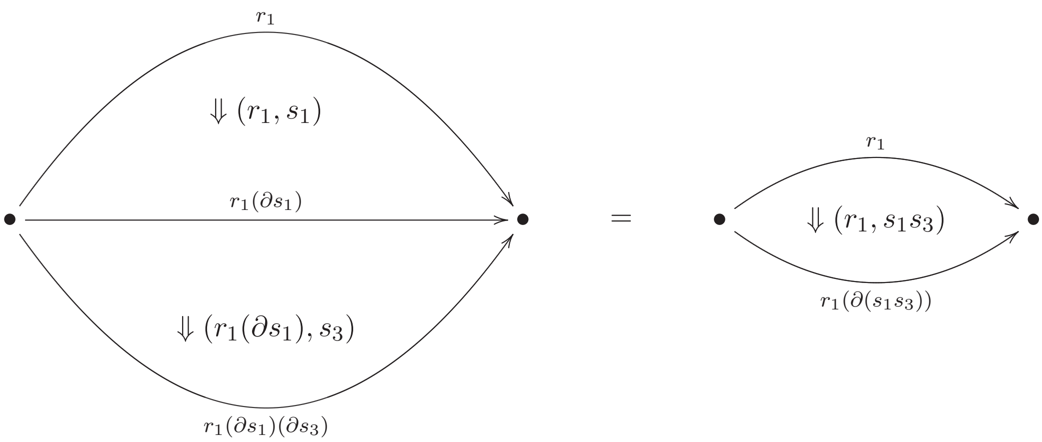
Horizontal composition $(r_1, s_1) \sharp_0 (r_2, s_2)$ of 2-cells is given by



There is a unique *horizontal identity* 2-cell

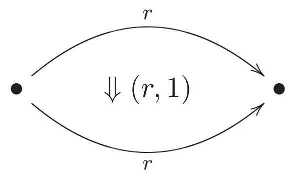


Similarly, when $r_1(\partial s_1) = r_3$, vertical composition $(r_1, s_1) \sharp_1 (r_3, s_3)$ of 2-cells is given by



For each $r \in R$ there is a *vertical identity* 2-cell

$\Downarrow (r, 1) \;=$



such that

$$\Downarrow (r,1) \; \sharp_1 \; \Downarrow (r,s) \; \sharp_1 \; \Downarrow (r(\partial s),1) \;=\; \Downarrow (r,s).$$

The *horizontal inverse* and the *vertical right inverse* of $\Downarrow (r,s)$ are $\Downarrow (r^{-1},(s^{-1})^{r^{-1}})$ and $\Downarrow (r(\partial s), s^{-1})$ respectively.

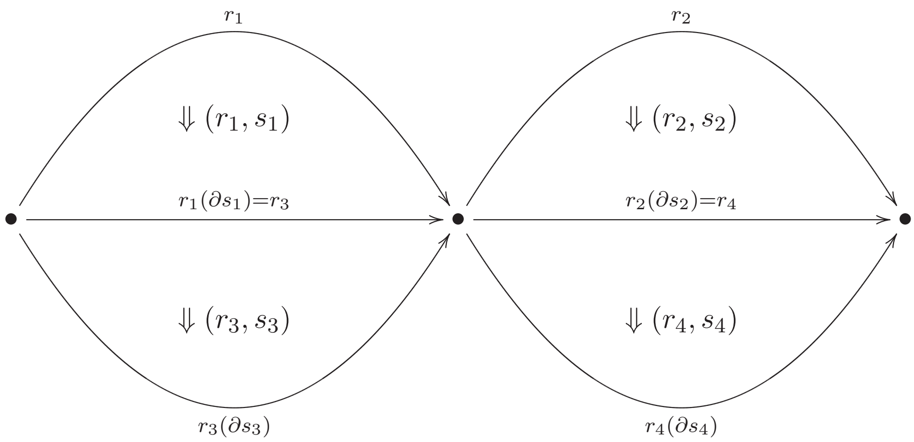
Horizontal composition with vertical identities is called *whiskering*. In diagrams it is often convenient to shrink $\Downarrow (q,1)$ to a single arc, labelled q, as in the whiskering formlae:

$$q_1 \, \sharp_0 \Downarrow (r,s) \; \sharp_0 \; q_2 \;=\; \Downarrow (q_1 r q_2, s^{q_2}).$$

The Peiffer condition for cat^1-groups establishes an *interchange law* for $\mathcal{H}$,

$$((r_1,s_1) \, \sharp_0 \, (r_2,s_2)) \, \sharp_1 \, ((r_3,s_3) \, \sharp_0 \, (r_4,s_4)) \quad = \quad ((r_1,s_1) \, \sharp_1 \, (r_3,s_3)) \, \sharp_0 \, ((r_2,s_2) \, \sharp_1 \, (r_4,s_4))$$

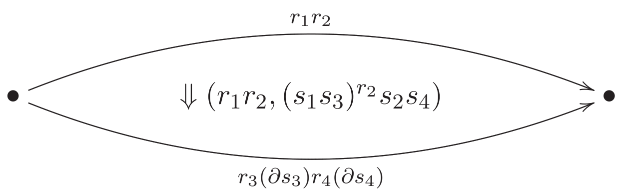
for the well-defined composite when $r_1(\partial s_1) = r_3$ and $r_2(\partial s_2) = r_4$,



When this composite is defined,

$$s_2 {s_3}^{r_4} \;=\; s_2 {s_3}^{r_2(\partial s_2)} \;=\; {s_3}^{r_2} s_2$$

and the composite 2-cell is



2 Derivations and Sections

2.1 Derivations

The Whitehead monoid $\mathrm{Der}(\mathcal{X})$ of $\mathcal{X}$ was defined in [54] to be the monoid of all *derivations* from R to S, that is the set of all maps $R \to S$, with composition $\star$, satisfying

$$\begin{array}{lrcl} \textbf{D1:} & \chi(qr) & = & (\chi q)^{r}\,(\chi r) \\ \textbf{D2:} & (\chi_1 \star \chi_2)(r) & = & (\chi_2 r)(\chi_1 r)(\chi_2 \partial \chi_1 r). \end{array}$$

The definition of Whitehead multiplication used here differs from that in [2] in that it is now defined as multiplication on the right rather than on the left, which is why we are using '$\star$' in place of '$\circ$'. Invertible elements in the monoid are called *regular*. The Whitehead group $W = W(\mathcal{X})$ is the group of the monoid.

In Brown and Gilbert [7] the notion of derivation was extended to that of γ-derivation, as in the following definition. Since ordinary derivations may be obtained from γ-derivations by setting γ to be the identity automorphism oh $\mathcal{X}$, we shall give properties in terms of the more general case.

Definition 2.1 *If $\gamma = (\ddot{\gamma}, \dot{\gamma})$ is an automorphism of $\mathcal{X}$, the Whitehead monoid $\mathrm{Der}_\gamma(\mathcal{X})$ of $\mathcal{X}$ is the monoid of all gamma-derivations from R to S, that is the set of all maps $R \to S$, with composition written $\star_\gamma$, satisfying*

$$\begin{array}{lrcl} \textbf{D1:} & \chi(qr) & = & (\chi q)^{\dot{\gamma} r}(\chi r)\ . \\ \textbf{D2:} & (\chi_1 \star_\gamma \chi_2)(r) & = & (\chi_2 r)(\chi_1 r)(\chi_2 \dot{\gamma}^{-1} \partial \chi_1 r). \end{array}$$

The following Lemma verifies that $\mathrm{Der}_\gamma(\mathcal{X})$ *is* a monoid.

Lemma 2.2

(a) $\chi 1 = 1$,

(b) $(\chi r)^{-1} \;=\; (\chi\, r^{-1})^{\dot{\gamma} r}$,

(c) *the zero map is a derivation and an identity for the Whitehead multiplication,*

(d) *the Whitehead multiplication is associative.*

Proof:

(a) This follows from $\chi(r1) = (\chi r)^1(\chi 1)$.

(b) This follows from $1 = \chi(r^{-1}r) = (\chi r^{-1})^{\dot{\gamma} r}(\chi r)$.

(c) It is clear that $0 : R \to S,\ r \mapsto 1$ is a derivation, and that

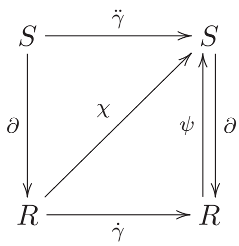
$$(\chi \star_\gamma 0)r = 1(\chi r)1 = \chi r = (\chi r)11 = (0 \star_\gamma \chi)r\ .$$

(d) Expansion by **D2:** using either bracketing (though one requires more work!) gives:

$$(\chi_1 \star \chi_2 \star \chi_3)r \;=\; (\chi_3 r)(\chi_2 r)(\chi_3 \dot{\gamma}^{-1} \partial \chi_2 r)(\chi_1 r)(\chi_3 \dot{\gamma}^{-1} \partial \chi_1 r)(\chi_2 \dot{\gamma}^{-1} \partial \chi_1 r)(\chi_3 \dot{\gamma}^{-1} \partial \chi_2 \dot{\gamma}^{-1} \partial \chi_1 r)\ .$$

□

For χ a γ-derivation, define $\psi = \psi_\chi : R \to S$ by $\psi r = \chi\dot{\gamma}^{-1}r$ or, equivalently, $\psi\dot{\gamma}r = \chi r$.



(11)

Then ψ is a (identity-) derivation since

$$\psi(qr) \;=\; \chi((\dot{\gamma}^{-1}q)(\dot{\gamma}^{-1}r)) \;=\; (\chi\dot{\gamma}^{-1}q)^{\dot{\gamma}(\dot{\gamma}^{-1}r)}(\chi\dot{\gamma}^{-1}r) \;=\; (\psi q)^r(\psi r).$$

Lemma 2.3 *The map* $\mathrm{Der}_\gamma(\mathcal{X}) \to \mathrm{Der}(\mathcal{X}),\ \chi \mapsto \psi_\chi$*, is a monoid homomorphism.*

Proof: If ψ_1, ψ_2 are the derivations corresponding to γ-derivations χ_1, χ_2, then

$$\begin{aligned}(\psi_1 \star \psi_2)(\dot{\gamma}r) &= (\psi_2\dot{\gamma}r)(\psi_1\dot{\gamma}r)(\psi_2\dot{\gamma}\dot{\gamma}^{-1}\partial\psi_1\dot{\gamma}r)\\ &= (\chi_2 r)(\chi_1 r)(\chi_2\dot{\gamma}^{-1}\partial\chi_1 r)\\ &= (\chi_1 \star_\gamma \chi_2)r.\end{aligned}$$

□

Lemma 2.4 *Given a γ-derivation χ of $\mathcal{X}$ there is an endomorphism $\beta_\chi = (\ddot{\beta}_\chi, \dot{\beta}_\chi)$ of $\mathcal{X}$ where*

$$\ddot{\beta}_\chi : S \to S,\ \ s \mapsto (\ddot{\gamma}s)(\chi\partial s), \quad \dot{\beta}_\chi : R \to R,\ \ r \mapsto (\dot{\gamma}r)(\partial\chi r)$$

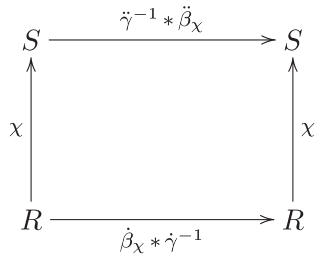
such that

(a) $\ddot{\beta}_\chi(s^r) \;=\; (\ddot{\beta}_\chi s)^{\dot{\beta}_\chi r} \;=\; (\chi r)^{-1}\,(\ddot{\beta}_\chi s)^{\dot{\gamma}r}\,(\chi r)$,

(b) $\dot{\beta}_\chi(q^r) \;=\; (\dot{\beta}_\chi q)^{\dot{\beta}_\chi r} \;=\; ((\dot{\gamma}r)(\partial\chi r))^{-1}(\dot{\gamma}q)(\partial\chi q)((\dot{\gamma}r)(\partial\chi r))$,

(c) $(\chi_1 \star_\gamma \chi_2)r \;=\; (\chi_2 r)(\ddot{\beta}_{\chi_2}\ddot{\gamma}^{-1}\chi_1 r) \;=\; (\chi_1 r)(\chi_2\dot{\gamma}^{-1}\dot{\beta}_{\chi_1} r)$,

(d) $\chi * \ddot{\gamma}^{-1} * \ddot{\beta}_\chi \;=\; \dot{\beta}_\chi * \dot{\gamma}^{-1} * \chi \;:\; R \to S, \quad r \mapsto (\chi r)(\chi\dot{\gamma}^{-1}\partial\chi r)$, so that the following diagram commutes:



(12)

(e) *The endomorphism* $\ddot{\beta}_\chi * \ddot{\gamma}^{-1}$ *commutes with* $\partial * \chi * \ddot{\gamma}^{-1}$ *while* $\dot{\gamma}^{-1} * \dot{\beta}_\chi$ *commutes with* $\dot{\gamma}^{-1} * \chi * \partial$.

(f) *When* $\psi = \psi_\chi$ *as in* (11)*, then* $\dot{\beta}_\chi r \;=\; \dot{\beta}_\psi(\dot{\gamma}r)$ *and* $\ddot{\beta}_\chi s \;=\; \ddot{\beta}_\psi(\ddot{\gamma}s)$.

Proof: We first check that $\dot\beta_\chi$ and $\ddot\beta_\chi$ are homomorphisms.

$$\begin{array}{rcl} \dot\beta_\chi(r_1r_2) &=& \dot\gamma(r_1r_2)\partial((\chi r_1)^{\dot\gamma r_2}(\chi r_2)) \;=\; (\dot\gamma r_1)(\partial\chi r_1)(\dot\gamma r_2)(\partial\chi r_2) \;=\; (\dot\beta_\chi r_1)(\dot\beta_\chi r_2), \\ \ddot\beta_\chi(s_1s_2) &=& \ddot\gamma(s_1s_2)(\chi((\partial s_1)(\partial s_2)) \;=\; (\ddot\gamma s_1)(\ddot\gamma s_2)(\chi\partial s_1)^{\partial\ddot\gamma s_2}(\chi\partial s_2) \;=\; (\ddot\beta s_1)(\ddot\beta s_2). \end{array}$$

We now verify the six properties.

(a) $$\begin{array}{rcl} \ddot\beta_\chi(s^r) &=& (\ddot\gamma s^r)(\chi\partial(s^r)) \;=\; (\ddot\gamma s)^{\dot\gamma r}(\chi(r^{-1}(\partial s)r)) \;=\; (\ddot\gamma s)^{\dot\gamma r}(\chi(r^{-1}))^{(\partial\ddot\gamma s)(\dot\gamma r)}(\chi\partial s)^{\dot\gamma r}(\chi r) \\ &=& \{(\chi(r^{-1}))(\ddot\gamma s)(\chi\partial s)\}^{\dot\gamma r}(\chi r) \;=\; (\chi r)^{-1}(\ddot\beta_\chi s)^{\dot\gamma r}(\chi r) \;=\; (\ddot\beta_\chi s)^{(\dot\gamma r)(\partial\chi r)} \;=\; (\ddot\beta_\chi s)^{\dot\beta_\chi r}\,. \end{array}$$

(b) $$\begin{array}{rcl} \dot\beta_\chi(q^r) &=& (\dot\gamma(q^r))(\partial\chi(r^{-1}qr)) \;=\; (\dot\gamma(q^r))\,\partial\{(\chi(r^{-1}))^{\dot\gamma(qr)}(\chi q)^{\dot\gamma r}(\chi r)\} \\ &=& (\dot\gamma(q^r))(\partial\chi(r^{-1}))^{(\dot\gamma r)(\dot\gamma q^r)}(\partial\chi q)^{\dot\gamma r}(\partial\chi r) \;=\; (\partial((\chi r)^{-1})(\dot\gamma(q^r))(\partial\chi q)^{\dot\gamma r}(\partial\chi r) \\ &=& (\partial\chi r)^{-1}\{(\dot\gamma q)(\partial\chi q)\}^{\dot\gamma r}(\partial\chi r) \;=\; (\dot\beta_\chi q)^{\dot\beta_\chi r}\,. \end{array}$$

(c) $$\begin{array}{rcl} (\chi_1 \star_\gamma \chi_2)r &=& (\chi_2 r)\{(\chi_1 r)(\chi_2\partial(\ddot\gamma^{-1}\chi_1 r))\} \;=\; (\chi_2 r)(\ddot\beta_{\chi_2}\ddot\gamma^{-1}\chi_1 r)\,, \\ (\chi_2\dot\gamma^{-1}\dot\beta_{\chi_1})r &=& \chi_2(r(\dot\gamma^{-1}\partial\chi_1 r)) \;=\; (\chi_2 r)^{\partial\chi_1 r}(\chi_2\dot\gamma^{-1}\partial\chi_1 r) \;=\; (\chi_1 r)^{-1}(\chi_2 r)(\chi_1 r)(\chi_2\dot\gamma^{-1}\partial\chi_1 r)\,. \end{array}$$

(d) $$\begin{array}{rcl} \ddot\beta_\chi(\ddot\gamma^{-1}\chi r) &=& \ddot\gamma(\ddot\gamma^{-1}\chi r)(\chi\partial\ddot\gamma^{-1}\chi r) \;=\; (\chi r)(\chi\dot\gamma^{-1}\partial\chi r)\,, \quad \text{and} \\ (\chi\dot\gamma^{-1})(\dot\beta_\chi r) &=& \chi(r(\dot\gamma^{-1}\partial\chi r)) \;=\; (\chi r)^{\partial\chi r}(\chi\dot\gamma^{-1}\partial\chi r) \;=\; (\chi r)(\chi\dot\gamma^{-1}\partial\chi r)\,. \end{array}$$

(e) By (d), $$\begin{array}{rcl} (\ddot\beta_\chi * \ddot\gamma^{-1}) * (\partial * \chi * \ddot\gamma^{-1}) &=& \partial * (\dot\beta_\chi * \dot\gamma^{-1} * \chi) * \ddot\gamma^{-1} \;=\; (\partial * \chi * \ddot\gamma^{-1}) * (\ddot\beta_\chi * \ddot\gamma^{-1}), \\ (\dot\gamma^{-1} * \dot\beta_\chi) * (\dot\gamma^{-1} * \chi * \partial) &=& \dot\gamma^{-1} * (\chi * \ddot\gamma^{-1} * \ddot\beta_\chi) * \partial \;=\; (\dot\gamma^{-1} * \chi * \partial) * (\dot\gamma^{-1} * \dot\beta_\chi). \end{array}$$

(f) This relationship between β_χ and β_ψ is immediate. □

Using Lemma 2.4 and the first crossed module axiom, the identity **D1:** for derivations generalises as follows.

Lemma 2.5

(a) $\chi(r_1r_2\ldots r_k) \;=\; (\chi r_1)^{\dot\gamma(r_2\ldots r_k)}(\chi r_2)^{\dot\gamma(r_3\ldots r_k)}\ldots(\chi r_{k-1})^{\dot\gamma r_k}(\chi r_k)\,,$

(b) $\partial\chi(r_1r_2\ldots r_k) \;=\; (\dot\gamma(r_1r_2\ldots r_k))^{-1}(\dot\beta_\chi r_1)(\dot\beta_\chi r_2)\ldots(\dot\beta_\chi r_k)\,,$

(c) $\chi\partial(s_1s_2\ldots s_k) \;=\; (\ddot\gamma(s_1s_2\ldots s_k))^{-1}(\ddot\beta_\chi s_1)(\ddot\beta_\chi s_2)\ldots(\ddot\beta_\chi s_k)\,.$

It is straightforward to verify that for g an invertible element in a monoid M, the set $M_g = (M, *_g)$ of elements in M with multiplication $*_g$ defined in terms of the usual multiplication by

$$m *_g n \;:=\; mg^{-1}n, \tag{13}$$

is a monoid with identity g. If $m \in M$ is invertible in M then m has $*_g$-inverse $\overline{m} := gm^{-1}g$. The resulting monoids are isomorphic, either by $\theta_g : M \to M_g, m \mapsto mg$ or by $\theta'_g : M \to M_g, m \mapsto gm$. When M is a group the g-conjugation automorphisms are the

$$\wedge_g m \;:\; G \to G,\; n \;\mapsto\; \overline{m} *_g n *_g m = gm^{-1}ng^{-1}m\,. \tag{14}$$

This notion generalises to categories and to crossed modules, but the application we require here is to the monoid of endomorphisms $\mathrm{End}_\gamma(\mathcal{X})$, where $\gamma = (\ddot\gamma, \dot\gamma)$ is an automorphism of $\mathcal{X}$, with multiplication

$$\alpha *_\gamma \beta \;:=\; (\ddot\alpha *_{\ddot\gamma} \ddot\beta,\; \dot\alpha *_{\dot\gamma} \dot\beta). \tag{15}$$

Theorem 2.6 *There is a monoid homomorphism* $\Delta_\gamma : \mathrm{Der}_\gamma(\mathcal{X}) \to \mathrm{End}_\gamma(\mathcal{X}),\ \chi \mapsto \beta_\chi = (\ddot\beta_\chi, \dot\beta_\chi)$.

Proof: Since
$$\begin{aligned}
(\ddot\beta_{\chi_1} *_\gamma \ddot\beta_{\chi_2})s &= (\ddot\beta_{\chi_1} * \ddot\gamma^{-1} * \ddot\beta_{\chi_2})s \\
&= \ddot\beta_{\chi_2}(s(\ddot\gamma^{-1}\chi_1\partial s) \\
&= (\ddot\gamma s)(\chi_1\partial s)\chi_2((\partial s)(\dot\gamma^{-1}\partial\chi_1\partial s) \\
&= (\ddot\gamma s)(\chi_1\partial s)(\chi_2\partial s)^{\partial\chi_1\partial s}(\chi_2\dot\gamma^{-1}\partial\chi_1\partial s) \\
&= (\ddot\gamma s)(\chi_2\partial s)(\chi_1\partial s)(\chi_2\dot\gamma^{-1}\partial\chi_1\partial s) \\
&= \ddot\beta_{\chi_1 \star_\gamma \chi_2}\ s, \\
(\dot\beta_{\chi_1} *_\gamma \dot\beta_{\chi_2})r &= (\dot\beta_{\chi_1} * \dot\gamma^{-1} * \dot\beta_{\chi_2})r \\
&= \dot\beta_{\chi_2}(r(\dot\gamma^{-1}\partial\chi_1 r)) \\
&= (\dot\gamma r)(\partial\chi_1 r)\partial\left((\chi_2 r)^{\partial\chi_1 r}(\chi_2\dot\gamma^{-1}\partial\chi_1 r)\right) \\
&= (\dot\gamma r)\partial\left((\chi_2 r)(\chi_1 r)(\chi_2\dot\gamma^{-1}\partial\chi_1 r)\right) \\
&= \dot\beta_{\chi_1\star\chi_2}\ r,
\end{aligned}$$
it follows that $(\Delta_\gamma\chi_1) *_\gamma (\Delta_\gamma\chi_2) = \Delta_\gamma(\chi_1 \star_\gamma \chi_2)$. □

We shall see later that there is a homomorphism from S to $\mathrm{Der}(\mathcal{X})$ mapping s to the principal derivation η_s.

Lemma 2.7 *For each* $s \in S$ *the function* η_s *(which we may also write as* $\eta_{\gamma,s}$ *when required)*
$$\eta_s : R \to S, \quad r \mapsto (s^{-1})^{\dot\gamma r} s$$
is a γ*-derivation, called a* principal γ-derivation*, satisfying*
$$\eta_s(\partial s_0) = [\ddot\gamma s_0, s], \qquad \textit{and} \qquad \partial(\eta_s r) = [\dot\gamma r, \partial s].$$

Proof:
$$(\eta_s q)^{\dot\gamma r}\,(\eta_s r) = ((s^{-1})^{\dot\gamma q}\,s)^{\dot\gamma r}\,(s^{-1})^{\dot\gamma r}\,s) = (s^{-1})^{\dot\gamma(qr)}\,s = \eta_s(qr)\ ,$$
$$\eta_s\partial s_0 = (s^{-1})^{\dot\gamma\partial s_0}\,s = (s^{-1})^{\partial\ddot\gamma s_0}\,s = (\ddot\gamma s_0)^{-1}\,s^{-1}\,(\ddot\gamma s_0)\,s = [\ddot\gamma s_0, s]\ ,$$
$$\partial(\eta_s r) = (\partial s^{-1})^{\dot\gamma r}(\partial s) = [\dot\gamma r, \partial s]\ .$$
□

Lemma 2.8 (Properties of principal derivations)

(a) η_1 *is the zero map,*

(b) $\ddot\beta_{\eta_s} s_0 = (\ddot\gamma s_0)^s$ *and* $\dot\beta_{\eta_s} r = (\dot\gamma r)^{\partial s}$,

(c) $\eta_{s_1} \star_\gamma \eta_{s_2} = \eta_{s_1 s_2}$ *and* $\overline{\eta_s} = \eta_{s^{-1}}$.

Proof:

(a) $\eta_1 r = 1^r 1 = 1$,

(b) $\ddot\beta_{\eta_s} s_0 = (\ddot\gamma s_0)(\eta_s(\partial s_0)) = (\ddot\gamma s_0)[\ddot\gamma s_0, s] = (\ddot\gamma s_0)^s$
and $\dot\beta_{\eta_s} r = (\dot\gamma r)(\partial\eta_s r) = (\dot\gamma r)[\dot\gamma r, \partial s] = (\dot\gamma r)^{\partial s}$,

(c) $(\eta_{s_2} r)(\eta_{s_1} r)(\eta_{s_2}\partial\ddot\gamma^{-1}\eta_{s_1} r)) = (\eta_{s_2} r)(\eta_{s_1} r)[\eta_{s_1} r, s_2] = (\eta_{s_2} r)s_2^{-1}(\eta_{s_1} r)s_2 = (s_2^{-1})^{\dot\gamma r}(s_1^{-1})^{\dot\gamma r} s_1 s_2$.
□

Lemma 2.9 *The following statements are equivalent.*

(i) χ *has a Whitehead* γ*-inverse* $\overline{\chi}$*;*

(ii) $\ddot{\beta}_\chi \in \mathrm{Aut}(S)$*, where* $\ddot{\beta}_\chi(s) = (\ddot{\gamma}s)(\chi\partial s)$*;*

(iii) $\dot{\beta}_\chi \in \mathrm{Aut}(R)$*, where* $\dot{\beta}_\chi(r) = (\dot{\gamma}r)(\partial\chi r)$*;*

(iv) $\beta = (\ddot{\beta}, \dot{\beta}) \in \mathrm{Aut}_\gamma(\mathcal{X})$.

When these conditions are satisfied,

$$\overline{\chi}r \;=\; (\ddot{\gamma}\overline{\ddot{\beta}_\chi}\,\chi\, r)^{-1} \;=\; (\chi\,\overline{\dot{\beta}_\chi}\dot{\gamma}\, r)^{-1}, \quad (\chi r)(\overline{\chi}r) \;=\; (\chi\dot{\gamma}^{-1}\partial\overline{\chi}r)^{-1}, \quad and \;\; (\overline{\chi}r)(\chi r) \;=\; (\overline{\chi}\dot{\gamma}^{-1}\partial\chi r)^{-1}.$$

Proof: Theorem 2.6 shows that, when χ is a regular derivation, both $\ddot{\beta}_\chi$ and $\dot{\beta}_\chi$ are automorphisms, so (i) implies (ii) and (iii), and hence (iv).

Now suppose that $\ddot{\beta}_\chi$ has γ-inverse $\overline{\ddot{\beta}_\chi}$. We first show that $\chi^\$$ is a derivation where $\chi^\$ r = (\ddot{\gamma}\overline{\ddot{\beta}_\chi}\chi r)^{-1}$. Using the equivalent formula, $\ddot{\beta}_\chi\ddot{\gamma}^{-1}\chi^\$ r = (\chi r)^{-1}$,

$$\begin{aligned}
\ddot{\beta}_\chi\ddot{\gamma}^{-1}(\,(\chi^\$ q)^{\dot{\gamma}r}\,(\chi^\$ r)\,) &= (\ddot{\beta}_\chi\ddot{\gamma}^{-1}\chi^\$ q)^{\dot{\beta}_\chi r}(\ddot{\beta}_\chi\ddot{\gamma}^{-1}\chi^\$ r) \quad \text{by Theorem 2.4 (a)}\\
&= ((\chi q)^{-1})^{(\dot{\gamma}r)(\partial\chi r)}(\chi r)^{-1} \qquad \text{by definition of} \quad \chi^\$\\
&= (\chi r)^{-1}(\,(\chi q)^{-1}\,)^{\dot{\gamma}r} \;=\; ((\chi q)^{\dot{\gamma}r}\,(\chi r))^{-1} \;=\; (\chi(qr))^{-1} \;=\; \ddot{\beta}_\chi\ddot{\gamma}^{-1}\chi^\$(qr)\,.
\end{aligned}$$

We now show that $\chi^\$$ is the Whitehead γ-inverse $\overline{\chi}$ of χ, using Lemma 2.4 (c), (d) :

$$\begin{aligned}
(\chi^\$ \star_\gamma \chi)r &= (\chi r)(\ddot{\beta}_\chi\ddot{\gamma}^{-1}\chi^\$ r) \;=\; (\chi r)(\chi r)^{-1}\,,\\
(\chi \star_\gamma \chi^\$)r &= (\chi r)(\chi^\$\dot{\gamma}^{-1}\dot{\beta}_\chi r) \;=\; (\chi r)(\ddot{\gamma}\overline{\ddot{\beta}_\chi}\chi\dot{\gamma}^{-1}\dot{\beta}_\chi r)^{-1} \;=\; (\chi r)(\chi r)^{-1}\,.
\end{aligned}$$

Thus (ii) implies (i), (iii) and (iv).

Similarly, suppose that $\dot{\beta}_\chi$ has an inverse $\dot{\beta}_\chi^{-1}$. We show that $\chi^\#$ is a derivation where $\chi^\# r = (\chi\overline{\dot{\beta}_\chi}\dot{\gamma}r)^{-1}$. Define $r' = \overline{\dot{\beta}_\chi}\dot{\gamma}r$ so that $\chi^\# r = (\chi r')^{-1}$, and similarly for q'. Then

$$\begin{aligned}
(\chi^\# q)^{\dot{\gamma}r}\,(\chi^\# r) &= \left((\chi q')^{-1}\right)^{\dot{\gamma}r}\,(\chi r')^{-1} \;=\; \Big((\chi r')(\chi q')^{\dot{\beta}_\chi r'}\Big)^{-1} \;=\; \Big((\chi r')(\chi q')^{(\dot{\gamma}r')(\partial\chi r')}\Big)^{-1}\\
&= \Big((\chi q')^{\dot{\gamma}r'}(\chi r')\Big)^{-1} \;=\; (\chi(qr)')^{-1} \;=\; \chi^\#(qr)\,.
\end{aligned}$$

This $\chi^\#$ is another form of $\overline{\chi}$ since, again using Lemma 2.4 (c), (d) :

$$\begin{aligned}
(\chi^\# \star_\gamma \chi)r &= (\chi r)(\ddot{\beta}_\chi\ddot{\gamma}^{-1}\chi^\# r) \;=\; (\chi r)(\ddot{\beta}_\chi\ddot{\gamma}^{-1}\chi\overline{\dot{\beta}_\chi}\dot{\gamma}r)^{-1} \;=\; (\chi r)(\chi r)^{-1},\\
(\chi \star_\gamma \chi^\#)r &= (\chi r)(\chi^\#\dot{\gamma}^{-1}\dot{\beta}_\chi r) \;=\; (\chi r)(\chi r)^{-1}\,.
\end{aligned}$$

Thus (iii) implies (i), (ii) and (iv).

Finally, (iv) implies (ii) and (iii), and hence (i).

The expressions for $(\chi r)(\overline{\chi}r)$ and $(\overline{\chi}r)(\chi r)$ are obtained by expanding $(\overline{\chi} \star_\gamma \chi)r$ and $(\chi \star_\gamma \overline{\chi})r$. □

We shall see in Subsection 3.2 that $W_\gamma(\mathcal{X})$ is the source group in the γ-*actor* of $\mathcal{X}$,

$$\mathrm{Act}_\gamma(\mathcal{X}) \;=\; (\Delta_\gamma : W_\gamma(\mathcal{X}) \to \mathrm{Aut}_\gamma(\mathcal{X}))\,.$$

Lue and Norrie, in [43, 48, 47], showed that $\mathrm{Act}(\mathcal{X})$ is the automorphism object of $\mathcal{X}$ in the category **XMod**. Gilbert, in [34], has discussed a connection between derivations and group extensions.

2.2 Sections

The construction for a cat[1]-group $\mathcal{C} = (e; t, h : G \to R)$ equivalent to the γ-derivation of the corresponding crossed module is the γ*-section*, namely a group monomorphism $\xi : R \to G$ satisfying:

$$\textbf{S1:} \quad t\xi(r) = \dot{\gamma}r \text{ for all } r \in R.$$

The equations

$$\xi r = (e\dot{\gamma}r)(\epsilon\chi r) = (\dot{\gamma}r, \chi r)\,, \qquad \chi r = (e\dot{\gamma}r)^{-1}(\xi r) \tag{16}$$

define a section ξ of $\mathcal{C}$ in terms of a derivation χ of $\mathcal{X}$, and conversely. The automorphism $\gamma = (\ddot{\gamma}, \dot{\gamma})$ of $\mathcal{X} = (\partial : S \to R)$ determines an automorphism $\bar{\gamma}$ of $R \ltimes S$, and hence an automorphism $(\bar{\gamma}, \dot{\gamma})$ of the corresponding cat[1]-group.

The *principal section* $\kappa_s,\ s \in \ker t,$ and the corresponding principal derivation η_s are given by

$$\eta_s r \;=\; (s^{-1})^{\dot{\gamma}r}s \qquad \kappa_s r \;=\; (e\dot{\gamma}r)^s \;=\; s^{-1}(e\dot{\gamma}r)s\,.$$

In the semidirect product notation we have

$$\kappa_s r \;=\; (\dot{\gamma}r, \eta_s r) \;=\; (\dot{\gamma}r, (s^{-1})^{\dot{\gamma}r}s) \;=\; (1, s^{-1})(\dot{\gamma}r, 1)(1, s) \;=\; (\dot{\gamma}r, 1)^{(1,s)}\,.$$

Since $(ehg^{-1})(\xi\dot{\gamma}^{-1}hg) \in \ker t$ and $(ehg^{-1})g \in \ker h$ we have, in the group groupoid,

$$g * \xi\dot{\gamma}^{-1}hg \;=\; g(ehg^{-1})(\xi\dot{\gamma}^{-1}hg) \;=\; (ehg)((ehg^{-1})g)((ehg^{-1})(\xi\dot{\gamma}^{-1}hg)) \;=\; (\xi\dot{\gamma}^{-1}hg)(ehg^{-1})g. \tag{17}$$

These sections form the monoid $\mathrm{Sect}(\mathcal{C})$ of $\mathcal{C}$, whose composition rule we determine from the rule **D2:** for $\mathrm{Der}(\mathcal{X})$ by evaluating:

$$\begin{aligned}
(\xi_1 \star_\gamma \xi_2)r &= (e\dot{\gamma}r)(\epsilon(\chi_1 \star \chi_2)r) \\
&= (e\dot{\gamma}r)(\epsilon\chi_2 r)(\epsilon\chi_1 r)(\epsilon\chi_2\dot{\gamma}^{-1}h\epsilon\chi_1 r) \\
&= (\xi_2 r)(e\dot{\gamma}r^{-1})(\xi_1 r)(eh(\epsilon\chi_1 r)^{-1})(\xi_2\dot{\gamma}^{-1}h\epsilon\chi_1 r) \\
&= (\xi_2 r)(e\dot{\gamma}r^{-1})(\xi_1 r)(eh((\xi_1 r)^{-1}(e\dot{\gamma}r)))(\xi_2\dot{\gamma}^{-1}h((e\dot{\gamma}r^{-1})(\xi_1 r))) \\
&= ((e\dot{\gamma}r)(\xi_2 r^{-1}))^{-1}((\xi_1 r)(eh\xi_1 r^{-1}))((e\dot{\gamma}r)(\xi_2 r^{-1}))(\xi_2\dot{\gamma}^{-1}h\xi_1 r).
\end{aligned}$$

Since $(e\dot{\gamma}r)(\xi_2 r^{-1}) \in \ker t$ while $(\xi_1 r)(eh\xi_1 r^{-1}) \in \ker h$, we obtain, using (17),

$$\textbf{S2:} \quad (\xi_1 \star_\gamma \xi_2)r \;=\; (\xi_1 r)(eh\xi_1 r^{-1})(\xi_2\dot{\gamma}^{-1}h\xi_1 r) \;=\; (\xi_2\dot{\gamma}^{-1}h\xi_1 r)(eh\xi_1 r^{-1})(\xi_1 r). \tag{18}$$

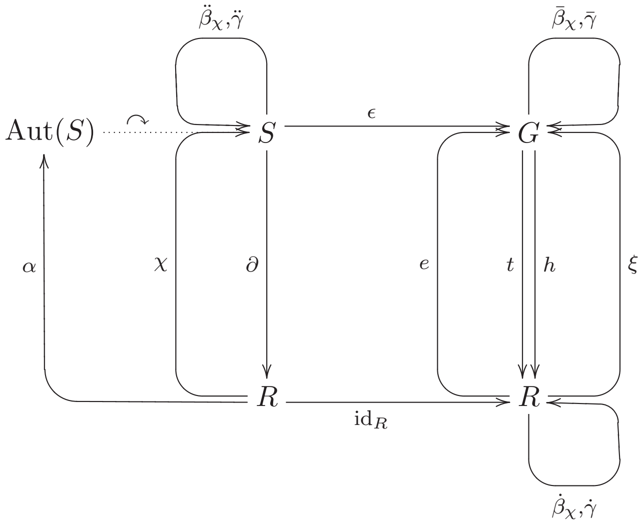
(Note that this axiom also differs from that in [2] in that it is converted to a multiplication on the right.)

The section $\dot{\gamma} * e$ is the identity for this composition, and equation (16) determines a monoid isomorphism $\mathrm{Der}(\mathcal{X}) \cong \mathrm{Sect}(\mathcal{C})$. A section is *regular* when $h\xi$ is an automorphism of R, and the group of regular sections is isomorphic to the Whitehead group.

Each χ and its associated ξ determine endomorphisms of $R, S, G, \mathcal{X}$ and $\mathcal{C}$, namely

$$\begin{aligned}
\dot{\beta}_\chi &= \dot{\beta}_\xi &: \quad & R \to R, \quad r \mapsto (\dot{\gamma}r)(\partial\chi r) = h\xi r, \\
\ddot{\beta}_\chi &= \ddot{\beta}_\xi &: \quad & S \to S, \quad s \mapsto (\ddot{\gamma}s)(\chi\partial s) \;=\; (\ddot{\gamma}s)(e\partial\ddot{\gamma}s^{-1})(\xi\partial s) \;=\; (\xi\partial s)(e\partial\ddot{\gamma}s^{-1})(\ddot{\gamma}s), \\
\bar{\beta}_\chi &= \bar{\beta}_\xi &: \quad & G \to G, \quad g \mapsto (eh\xi tg)(\xi tg^{-1})(\bar{\gamma}g)(eh\bar{\gamma}g^{-1})(\xi hg), \\
(\ddot{\beta}_\chi, \dot{\beta}_\chi) &= (\ddot{\beta}_\xi, \dot{\beta}_\xi) &: \quad & \mathcal{X} \to \mathcal{X}, \\
(\bar{\beta}_\chi, \dot{\beta}_\chi) &= (\bar{\beta}_\xi, \dot{\beta}_\xi) &: \quad & \mathcal{C} \to \mathcal{C},
\end{aligned} \tag{19}$$

and these assignments determine group homomorphisms from the Whitehead group to these five endomorphism groups. The accompanying diagram shows the relationship between the various groups and homomorphisms.



2.3 The group-groupoid equivalent of derivations and sections

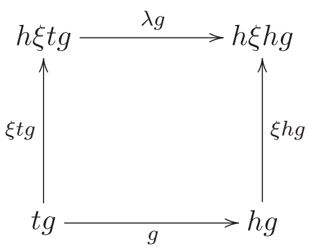
(This Subsection (for now) covers only identity derivations and sections.)

The cat^1−formula (18) for Whitehead composition of sections is

$$\textbf{S2:} \qquad (\xi_1 \star \xi_2)r \;=\; (\xi_1 r)(eh\xi_1 r^{-1})(\xi_2 h\xi_1 r) \;=\; (\xi_2 h\xi_1 r)(eh\xi_1 r^{-1})(\xi_1 r)\;,$$

which is rather obscure. Considering the group-groupoid $\mathcal{G}$ associated to the cat^1-group $\mathcal{C}$, as discussed in Subsection 1.13, we see that sections of $\mathcal{C}$ are associated to automorphisms of $\mathcal{G}$.

A section ξ of $\mathcal{C}$ defines a groupoid endomorphism $\lambda = \lambda_\xi : \mathcal{G} \to \mathcal{G}$ as follows. Consider the diagram



(20)

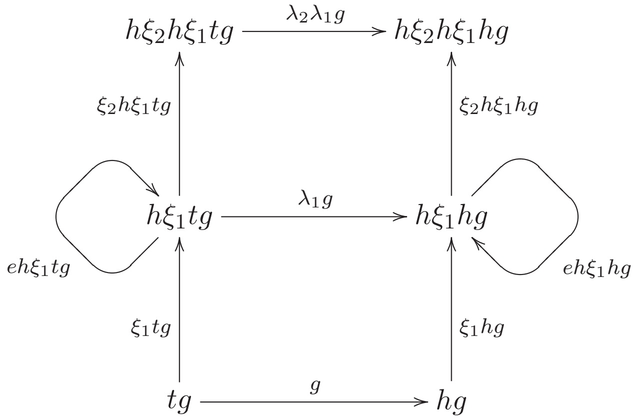
where $t\lambda g = h\xi tg$ and $h\lambda g = h\xi hg$. The morphism λ is defined on objects and arrows by

$$\lambda r \;=\; h\xi r, \qquad \lambda g \;=\; (\widetilde{\xi tg}) * g * \xi hg \;=\; (eh\xi tg)(\xi tg^{-1})g(ehg^{-1})(\xi hg). \tag{21}$$

The product of the first four terms is in $\ker h$, while the product of the last four terms is in $\ker t$. It is easily verified that λ is a groupoid morphism. If $r_0 = tg_1,\ r_1 = hg_1 = tg_2$ and $r_2 = hg_2$, then

$$\begin{aligned}
(\lambda g_1) * (\lambda g_2) &= (eh\xi r_0)(\xi r_0^{-1})g_1(er_1^{-1})(\xi r_1).(eh\xi r_1^{-1}).(eh\xi r_1)(\xi r_1^{-1})g_2(er_2^{-1})(\xi r_2) \\
&= (eh\xi r_0)(\xi r_0^{-1})(g_1 * g_2)(er_2^{-1})(\xi r_2) \\
&= \lambda(g_1 * g_2).
\end{aligned}$$

When we consider ξ_1 followed by ξ_2 we get



and the composite on the left-hand side is

$$(\xi_1 tg) * (\xi_2 h\xi_1 tg) \;=\; (\xi_1 tg)(eh\xi_1 tg^{-1})(\xi_2 h\xi_1 tg)$$

in agreement with **S2:**, and similarly for the right-hand side. Thus $\lambda_{\xi_1 \star \xi_2} \;=\; \lambda_{\xi_1} * \lambda_{\xi_2}$ and we have the following result.

Lemma 2.10 *There is a monoid homomorphism*

$$\mathrm{Sect}(\mathcal{C}) \to \mathrm{End}(\mathcal{G}), \qquad \xi \mapsto \lambda_\xi$$

which restricts to a homomorphism $W(\mathcal{C}) \to \mathrm{Aut}(\mathcal{G})$.

Associated to a principal derivation $\eta_s r = (s^{-1})^r s$, and the corresponding principal section $\kappa_s r = (er)^s$, there is a *principal endomorphism* λ_s of $\mathcal{G}$.

Proposition 2.11 *The principal endomorphism* $\lambda_s = \lambda_{\kappa_s} : \mathcal{G} \to \mathcal{G}$ *is given by*

$$\lambda_s r \;=\; r^{hs}, \qquad \lambda_s g \;=\; g^{hs}.$$

Proof: Applying the formulae in equation (21), and $[\ker t, \ker s] = 1$,

$$\begin{aligned}
\lambda_s r &= h\kappa_s r \;=\; (hs^{-1})r(hs) \;=\; r^{hs}, \\
\lambda_s g &= (\widetilde{\kappa_s tg}) * g * (\kappa_s hg) \\
&= (\widetilde{(etg)^s}) * g * (ehg)^s \\
&= (eh(s^{-1}(etg)s)(s^{-1}(etg)s)^{-1}(et(s^{-1}(etg)s)(etg^{-1})g(ehg)^{-1}(ehg)^s \\
&= (ehs^{-1})(etg)(ehs)s^{-1}(etg^{-1})[s][g(ehg)^{-1}]s^{-1}(ehg)s \\
&= (ehs^{-1})(etg)[(ehs)s^{-1}][(etg^{-1})g]s \\
&= (ehs^{-1})g(ehs) \\
&= g^{hs}.
\end{aligned}$$

□

3 The Actor of a Crossed Module

This section is based on the material covered in pages 25-28 of Norrie's thesis [47]. We will, however, be extending her actor crossed module $\mathrm{Act}(\mathcal{X})$ to the more general $\mathrm{Act}_\gamma(\mathcal{X})$ where γ is an automorphism of $\mathcal{X}$. Here is a table giving Norrie's symbols and the ones used here.

section	type	old symbol	new symbol
3	xmod	$\partial : T \to G$	$\mathcal{X} = (\partial : S \to R)$
	Whitehead group	$\mathrm{Der}(G,T)$	$W(\mathcal{X})$
	xmod morphism	(σ, θ)	$\beta_\chi \ = \ (\ddot{\beta}_\chi, \dot{\beta}_\chi) \ = \ (\sigma, \rho)$
	principal derivations	$E(G,T)$	$E(\mathcal{X})$
3.6	xmod	$\mu : M \to P$	$\mathcal{X} = (\partial : S \to R)$
	xmod	$\nu : N \to V$	$\mathcal{Y} = (\delta : Q \to P)$
	xmod morphism	$< \epsilon, \rho >$	$\beta \ (= \ (\ddot{\beta}, \dot{\beta})\,)$
	derivation group	$D(P,M)$	$W = W(\mathcal{X})$
	automorphism group	$\mathrm{Aut}(M,P)$	$A = A(\mathcal{X})$
	semidirect product	$(M,P) \sqsupset_{<\epsilon,\rho>} (N,V)$	$\mathcal{Y} \ltimes \mathcal{X}$
	xmod	$\pi \ : \ M \sqsupset N \ \to \ P \sqsupset V$	$(\pi \ : \ Q \ltimes S \to P \ltimes R)$
	automorphism	$\rho(v) = (\rho_1(v), \rho_2(v))$	$\beta(p) = \beta_p : \mathcal{X} \to \mathcal{X}$
	elements	$m,\ n,\ p,\ v$	$s,\ q,\ r,\ p$
	Whitehead boundary	$\Delta(\chi) \ = < \theta_\chi, \ddot{\beta}_\chi >$	$\Delta(\chi) \ = \ \beta_\chi \ = \ (\ddot{\beta}_\chi, \dot{\beta}_\chi)$

When γ is an automorphism of $\mathcal{X}$, the group of automorphisms $\mathrm{Aut}_\gamma\mathcal{X}$, has composition (as in (15)) given by

$$\alpha_1 *_\gamma \alpha_2 \ := \ (\ddot{\alpha}_1 *_{\ddot{\gamma}} \ddot{\alpha}_2,\ \dot{\alpha}_1 *_{\dot{\gamma}} \dot{\alpha}_2) \ = \ (\ddot{\alpha}_1 * \ddot{\gamma}^{-1} * \ddot{\alpha}_2,\ \dot{\alpha}_1 * \dot{\gamma}^{-1} * \dot{\alpha}_2)\,.$$

3.1 The Lue and Norrie crossed modules $\mathcal{L}_\gamma(\mathcal{X})$ and $\mathcal{N}_\gamma(\mathcal{X})$ of $\mathcal{X}$

We generalise the automorphism crossed module $(\iota : R \to \mathrm{Aut}\,R)$, where ιr is conjugation by r, to the *Norrie crossed module* $\mathcal{N}_\gamma = (i_\gamma : R \to \mathrm{Aut}_\gamma\mathcal{X})$ where:

- the γ-conjugation map is given by $i_\gamma r := \beta_r$ where $\dot{\beta}_r q := (\dot{\gamma}q)^r,\ \ddot{\beta}_r s := (\ddot{\gamma}s)^r$, and
- $\mathrm{Aut}_\gamma\mathcal{X}$ has right actions on R and S given by $r^\alpha \ := \ \dot{\alpha}\dot{\gamma}^{-1}r, \quad s^\alpha \ := \ \ddot{\alpha}\ddot{\gamma}^{-1}s\,.$

(An alternative set of definitions is given by $\dot{\beta}_r q = (\dot{\gamma}q)^r,\ \ddot{\beta}_r s = (\ddot{\gamma}s)^r,\ r^\alpha = \dot{\alpha}\dot{\gamma}^{-1}r,\ s^\alpha = \ddot{\alpha}\ddot{\gamma}^{-1}s$, but these do not combine with the principal derivation map to give a morphism of crossed modules.)
Note that β_r^{-1} is given by $\dot{\beta}_r^{-1}q = \dot{\gamma}^{-1}(q^{r^{-1}}),\ \ddot{\beta}_r^{-1}s = \ddot{\gamma}^{-1}(s^{r^{-1}})$.

We now check the various axioms for $\mathcal{N}_\gamma$.
The map i_γ is a homomorphism:

$$\begin{aligned}(\beta_{r_1} *_\gamma \beta_{r_2})\,q \ &= \ \dot{\beta}_{r_2}\dot{\gamma}^{-1}((\dot{\gamma}q)^{r_1}) \ = \ \dot{\beta}_{r_2}(q^{\dot{\gamma}^{-1}r_1}) \ = \ (\dot{\gamma}(q^{\dot{\gamma}^{-1}r_1}))^{r_2} \ = \ (\dot{\gamma}q)^{r_1r_2} \ = \ \dot{\beta}_{r_1r_2}q\,,\\ (\beta_{r_1} *_\gamma \beta_{r_2})\,s \ &= \ \ddot{\beta}_{r_2}\ddot{\gamma}^{-1}((\ddot{\gamma}s)^{r_1}) \ = \ \ddot{\beta}_{r_2}(s^{\dot{\gamma}^{-1}r_1}) \ = \ (\ddot{\gamma}(s^{\dot{\gamma}^{-1}r_1}))^{r_2} \ = \ (\ddot{\gamma}s)^{r_1r_2} \ = \ \ddot{\beta}_{r_1r_2}s\,.\end{aligned}$$

The given formulae do specify an action of $\mathrm{Aut}_\gamma\mathcal{X}$ on $\mathcal{X}$:

$$\begin{aligned}(r^{\alpha_1})^{\alpha_2} \ &= \ \dot{\alpha}_2\dot{\gamma}^{-1}(\dot{\alpha}_1\dot{\gamma}^{-1}r) \ = \ (\dot{\alpha}_1 *_{\dot{\gamma}} \dot{\alpha}_2)\dot{\gamma}^{-1}r \ = \ r^{\alpha_1 *_\gamma \alpha_2}\,,\\ (s^{\alpha_1})^{\alpha_2} \ &= \ \ddot{\alpha}_2\ddot{\gamma}^{-1}(\ddot{\alpha}_1\ddot{\gamma}^{-1}s) \ = \ (\ddot{\alpha}_1 *_{\ddot{\gamma}} \ddot{\alpha}_2)\ddot{\gamma}^{-1}s \ = \ s^{\alpha_1 *_\gamma \alpha_2}\,.\end{aligned}$$

First crossed module axiom (using the γ-conjugation of (14)):

$$\begin{aligned}(\wedge_\gamma\alpha)\dot\beta_r q &= \dot\alpha\dot\gamma^{-1}\dot\beta_r(\dot\alpha^{-1}\dot\gamma q) = \dot\alpha\dot\gamma^{-1}((\dot\gamma\dot\alpha^{-1}\dot\gamma q)^r) = \dot\alpha((\dot\alpha^{-1}\dot\gamma q)^{\dot\gamma^{-1}r}) = (\dot\gamma q)^{r^\alpha} = \dot\beta_{r^\alpha} q\,,\\ (\wedge_\gamma\alpha)\ddot\beta_r s &= \ddot\alpha\ddot\gamma^{-1}\ddot\beta_r(\ddot\alpha^{-1}\ddot\gamma s) = \ddot\alpha\ddot\gamma^{-1}((\ddot\gamma\ddot\alpha^{-1}\ddot\gamma s)^r) = \ddot\alpha((\ddot\alpha^{-1}\ddot\gamma s)^{\dot\gamma^{-1}r}) = (\ddot\gamma s)^{r^\alpha} = \ddot\beta_{r^\alpha} s\,.\end{aligned}$$

Second crossed module axiom:

$$r^{i_\gamma r'} = \dot\beta_{r'}\dot\gamma^{-1}r = r^{r'}\,.$$

Similarly, for the *Lue crossed module* $\mathcal{L}_\gamma(\mathcal{X}) = (\partial * i_\gamma : S \to \mathrm{Aut}_\gamma\mathcal{X})$,

- the boundary maps $s \in S$ to $\beta_{\partial s}$ where $\dot\beta_{\partial s}q = (\dot\gamma q)^{\partial s}$, $\ddot\beta_{\partial s}s' = (\ddot\gamma s')^{\partial s}$, and
- the action of $\mathrm{Aut}_\gamma\mathcal{X}$ on R and S are as above.

The verification of the crossed module axioms for $\mathcal{L}_\gamma\mathcal{X}$ are similar to those for $\mathcal{N}_\gamma\mathcal{X}$.

3.2 The actor crossed module $\mathcal{A}_\gamma(\mathcal{X})$ of $\mathcal{X}$

The missing part of the structure of the actor crossed module $\mathrm{Act}_\gamma(\mathcal{X})$ is a γ-action of the automorphisms on the derivations.

Lemma 3.1 *There is an action of* $\mathrm{Aut}_\gamma(\mathcal{X})$ *on* $W_\gamma(\mathcal{X})$ *given by*

$$\chi^\alpha \;=\; \gamma * \alpha^{-1} * \chi * \gamma^{-1} * \alpha \;:\; R \to S\,, \quad r \mapsto \ddot\alpha\ddot\gamma^{-1}\chi\dot\alpha^{-1}\dot\gamma\, r\,,$$

such that $\beta_{\chi^\alpha} = (\wedge_\gamma\alpha)(\beta_\chi)$ *where* $\wedge_\gamma\alpha$ *is the* γ*-conjugation automorphism of (14).*

Proof: We first check the axiom for an action:

$$(\chi^{\alpha_1})^{\alpha_2} \;=\; \gamma*\alpha_2^{-1}*(\gamma*\alpha_1^{-1}*\chi*\gamma^{-1}*\alpha_1)*\gamma^{-1}*\alpha_2 \;=\; \gamma*(\alpha_1*_\gamma\alpha_2)^{-1}*\chi*\gamma^{-1}*(\alpha_1*_\gamma\alpha_2) \;=\; \chi^{(\alpha_1*_\gamma\alpha_2)}.$$

Secondly, we observe that $(\wedge_\gamma\alpha)(\beta_\chi) = \gamma * \alpha^{-1} * \beta_\chi * \gamma^{-1} * \alpha$. □

Definition 3.2 *For* γ *an automorphism of* $\mathcal{X} = (\partial : S \to R)$*, the* actor crossed module over γ of $\mathcal{X}$ *is* $\mathcal{A}_\gamma(\mathcal{X}) = (\Delta_\gamma : W_\gamma \to A_\gamma)$ *where*

- $W_\gamma = W_\gamma(\mathcal{X})$ *is the Whitehead group of invertible derivations*

$$\chi : R \to S, \quad \textit{such that} \quad \chi(qr) = (\chi q)^{\dot\gamma r}\,(\chi r) \quad \textit{for all} \quad q, r \in R\,, \tag{22}$$

  and with Whitehead multiplication (on the right)

$$\chi_1 \star_\gamma \chi_2 \;:\; R \to S,\;\; r \mapsto (\chi_2 r)(\chi_1 r)(\chi_2\dot\gamma^{-1}\partial\chi_1 r)\;; \tag{23}$$

- $A_\gamma = \mathrm{Aut}_\gamma(\mathcal{X})$ *is the group of automorphisms of* $\mathcal{X}$*, namely those invertible* $\alpha = (\ddot\alpha, \dot\alpha) : \mathcal{X} \to \mathcal{X}$ *such that*

$$\dot\alpha\partial = \partial\ddot\alpha\,, \quad \ddot\alpha(s^r) = (\ddot\alpha s)^{\dot\alpha r} \quad \textit{and} \quad \dot\alpha(q^r) = (\dot\alpha q)^{\dot\alpha r} \quad \textit{for all}\;\; s \in S \;\; \textit{and} \;\; q, r \in\,,$$

  with composition $\alpha_1 *_\gamma \alpha_2 := (\ddot\alpha_1 *_{\ddot\gamma} \ddot\alpha_2,\, \dot\alpha_1 *_{\dot\gamma} \dot\alpha_2)$*, and action given by Lemma 3.1.*

- *The boundary map is obtained by restricting the monoid homomorphism* $\Delta_\gamma : \mathrm{Der}_\gamma(\mathcal{X}) \to \mathrm{End}_\gamma(\mathcal{X})$ *of Theorem 2.6 to the regular derivations:*

$$\Delta_\gamma \,:\, W_\gamma \to A_\gamma, \;\; \chi \mapsto \beta_\chi = (\ddot{\beta}_\chi, \dot{\beta}_\chi) \;,$$

$$\textit{where} \quad \ddot{\beta}_\chi : S \to S, \;\; s \mapsto (\ddot{\gamma}s)(\chi\partial s), \quad \dot{\beta}_\chi : R \to R, \;\; r \mapsto (\dot{\gamma}r)(\partial\chi r)\,. \tag{24}$$

When it is convenient not to distinguish the two group homomorphisms in the crossed module morphism, we write α for both $\ddot{\alpha}$ and $\dot{\alpha}$.

These groups and morphisms are exhibited in the following diagram (the inner morphism $\iota_\gamma = (\ddot{\iota}_\gamma, \dot{\iota}_\gamma)$ is defined in Subsection 3.3 below):

$$\begin{array}{ccccc} S & \xleftarrow{\;\ddot{\alpha},\, \ddot{\beta}_r,\, \ddot{\beta}_\chi,\, \ddot{\gamma}\;} & S & \xrightarrow{\;\ddot{\iota}_\gamma\;} & W \\ \Big\downarrow{\scriptstyle \partial} & & \Big\downarrow{\scriptstyle \partial} & & \Big\downarrow{\scriptstyle \Delta_\gamma} \\ R & \xleftarrow[\;\dot{\alpha},\, \dot{\beta}_r,\, \dot{\beta}_\chi,\, \dot{\gamma}\;]{} & R & \xrightarrow[\;\dot{\iota}_\gamma\;]{} & A \end{array} \tag{25}$$

Theorem 3.3 *With this action,* $\mathcal{A}_\gamma(\mathcal{X}) = (\Delta_\gamma : W_\gamma \to A_\gamma)$ *is a crossed module.*

Proof: We have already shown that Δ_γ is a group homomorphism.

We verify the first crossed module axiom for $\mathcal{A}_\gamma(\mathcal{X})$ as follows.

$$\textbf{X1:} \qquad \Delta_\gamma(\chi^\alpha) \;=\; (\wedge_\gamma\alpha)(\Delta_\gamma\chi) \;=\; \gamma * \alpha^{-1} * \beta_\chi * \gamma^{-1} * \alpha\,.$$

Put $\chi^\alpha \;=\; \chi^+$ and $\Delta_\gamma(\chi^+) \;=\; \alpha^+ \;=\; (\ddot{\alpha}^+, \dot{\alpha}^+)$. Then

$$\begin{aligned} \ddot{\alpha}^+ s \;=\; (\ddot{\gamma}s)(\chi^+\partial s) \;&=\; (\ddot{\gamma}s)(\ddot{\alpha}\ddot{\gamma}^{-1}\chi\dot{\alpha}^{-1}\dot{\gamma}\partial s) \;=\; (\ddot{\gamma}s)(\ddot{\alpha}\ddot{\gamma}^{-1}\chi\partial\ddot{\alpha}^{-1}\ddot{\gamma}s) \;=\; \ddot{\alpha}\ddot{\gamma}^{-1}((\ddot{\gamma}\ddot{\alpha}^{-1}\ddot{\gamma}s)(\chi\partial\ddot{\alpha}^{-1}\ddot{\gamma}s)) \\ &=\; \ddot{\alpha}\ddot{\gamma}^{-1}\ddot{\beta}_\chi(\ddot{\alpha}^{-1}\ddot{\gamma}s) \;=\; (\gamma * \alpha^{-1} * \beta_\chi * \gamma^{-1} * \alpha)\, s\;, \\ \dot{\alpha}^+ r \;=\; (\dot{\gamma}r)(\partial\chi^+ r) \;&=\; (\dot{\gamma}r)(\partial\ddot{\alpha}\ddot{\gamma}^{-1}\chi\dot{\alpha}^{-1}\dot{\gamma}r) \;=\; (\dot{\gamma}r)(\dot{\alpha}\dot{\gamma}^{-1}\partial\chi\dot{\alpha}^{-1}\dot{\gamma}r) \;=\; \dot{\alpha}\dot{\gamma}^{-1}(\dot{\gamma}\dot{\alpha}^{-1}\dot{\gamma}r)(\partial\chi\dot{\alpha}^{-1}\dot{\gamma}r)) \\ &=\; \dot{\alpha}\dot{\gamma}^{-1}\dot{\beta}_\chi(\dot{\alpha}^{-1}\dot{\gamma}r) \;=\; (\gamma * \alpha^{-1} * \beta_\chi * \gamma^{-1} * \alpha)\, r\;. \end{aligned}$$

The second crossed module axiom for $\mathcal{A}_\gamma(\mathcal{X})$,

$$\textbf{X2:} \qquad {\chi_1}^{\Delta_\gamma\chi_2} \;=\; \overline{\chi_2} \star_\gamma \chi_1 \star_\gamma \chi_2\,,$$

is verified by showing that $\chi_2 \star_\gamma {\chi_1}^{\Delta_\gamma\chi_2} = \chi_1 \star_\gamma \chi_2$, using Lemma 2.4 (c),

$$(\chi_2 \star_\gamma {\chi_1}^{\Delta_\gamma\chi_2})\, r \;=\; (\chi_2 r)({\chi_1}^{\Delta_\gamma\chi_2}\dot{\gamma}^{-1}\dot{\beta}_{\chi_2}r) \;=\; (\chi_2 r)(\ddot{\beta}_{\chi_2}\ddot{\gamma}^{-1}\chi_1 r) \;=\; (\chi_1 \star_\gamma \chi_2) r\,.$$

□

3.3 The inner morphism $\iota : \mathcal{X} \to \mathcal{A}(\mathcal{X})$

We next describe the morphism of crossed modules $\iota_\gamma = (\ddot{\iota}_\gamma, \dot{\iota}_\gamma) : \mathcal{X} \to \mathcal{A}_\gamma(\mathcal{X})$. The conditions in (1) for ι_γ to be a morphism are:

$$\ddot{\iota}_\gamma(s_1 s_2) \;=\; \ddot{\iota}_\gamma s_1 \star_\gamma \ddot{\iota}_\gamma s_2\,, \qquad \dot{\iota}_\gamma(r_1 r_2) \;=\; \dot{\iota}_\gamma r_1 *_\gamma \dot{\iota}_\gamma r_2\,. \tag{26}$$

The range part $\imath_\gamma$ of ι_γ is given in Subsection 3.1 by:

$$\imath_\gamma \,:\, R \to A_\gamma, \quad r \mapsto \beta_r = (\ddot{\beta}_r, \dot{\beta}_r) : \mathcal{X} \to \mathcal{X}, \quad \ddot{\beta}_r s_0 = (\ddot{\gamma} s_0)^r, \quad \dot{\beta}_r r_0 = (\dot{\gamma} r_0)^r \,.$$

The source part $\ddot{\imath}_\gamma$ of ι_γ maps s to its principal derivation (see Lemmas 2.7, 2.8):

$$\ddot{\imath}_\gamma \,:\, S \to W_\gamma, \quad s \mapsto \eta_s : R \to S, \; r \mapsto (s^{-1})^{\dot{\gamma} r}\, s\,.$$

Theorem 3.4 *The pair of group homomorphisms $\iota_\gamma = (\ddot{\imath}_\gamma, \imath_\gamma) \,:\, \mathcal{X} \to \mathcal{A}_\gamma(\mathcal{X})$ is a morphism of crossed modules.*

Proof: The square commutes if $\Delta_\gamma\, \ddot{\imath}_\gamma = \imath_\gamma\, \partial$. To verify this we show that $(\Delta_\gamma \ddot{\imath}_\gamma)s = \Delta_\gamma \eta_s = \beta_{\eta_s}$ is the same automorphism of $\mathcal{X}$ as $(\imath\partial)s = \beta_{\partial s}$. By definition of $\beta_{\eta_s} = (\ddot{\beta}_{\eta_s}, \dot{\beta}_{\eta_s})$ we have:

$$\begin{aligned}
\ddot{\beta}_{\eta_s}(s_0) &= (\ddot{\gamma} s_0)(\eta_s \partial s_0) = (\ddot{\gamma} s_0)(s^{-1})^{\partial \ddot{\gamma} s_0} s = s^{-1}(\ddot{\gamma} s_0) s = (\ddot{\gamma} s_0)^{\partial s} = \ddot{\beta}_{\partial s}(s_0)\,,\\
\dot{\beta}_{\eta_s}(r_0) &= (\dot{\gamma} r_0)(\partial \eta_s r_0) = (\dot{\gamma} r_0)\partial((s^{-1})^{\dot{\gamma} r_0} s) = (\partial s)^{-1}(\dot{\gamma} r_0)(\partial s) = (\dot{\gamma} r_0)^{\partial s} = \dot{\beta}_{\partial s}(r_0)\,.
\end{aligned}$$

Then we check that the action is preserved:

$$\begin{aligned}
(\ddot{\imath} s)^{\imath r}(q) &= (\eta_s)^{\beta_r} q = \ddot{\beta}_r \ddot{\gamma}^{-1} \eta_s \dot{\beta}_r^{-1} \dot{\gamma} q = \ddot{\beta}_r \ddot{\gamma}^{-1} \eta_s \dot{\gamma}^{-1} \left((\dot{\gamma} q)^{r^{-1}} \right) = \left(\ddot{\gamma}\ddot{\gamma}^{-1} \eta_s (q^{\dot{\gamma}^{-1} r^{-1}}) \right)^r \\
&= \left((s^{-1})^{r(\dot{\gamma} q) r^{-1}} s \right)^r = \left((s^r)^{-1} \right)^{\dot{\gamma} q} s^r = \eta_{(s^r)} q = \ddot{\imath}(s^r)(q)\,.
\end{aligned}$$

□

The γ-version of the *inner actor crossed module* of $\mathcal{X}$ is the image $\iota_\gamma \mathcal{X}$. The source group consists of the principal γ-derivations, and the range group consists of the γ-conjugation automorphisms. For further details see Norrie's thesis [47].

3.4 The Whitehead crossed module $\mathcal{W}_\gamma(\mathcal{X})$ of $\mathcal{X}$

Lemma 3.5 *There is an action of the Whitehead group W_γ on S given by*

$$s^\chi := s^{\beta_\chi} = \ddot{\beta}_\chi \ddot{\gamma}^{-1} s = s(\chi \dot{\gamma}^{-1} \partial s)$$

which makes $\mathcal{W}_\gamma(\mathcal{X}) = (\ddot{\imath}_\gamma : S \to W_\gamma)$ a crossed module.

Proof: We first verify that this is an action:

$$\begin{aligned}
(s^{\chi_1})^{\chi_2} &= \ddot{\beta}_{\chi_2} \ddot{\gamma}^{-1} \left(s(\chi_1 \dot{\gamma}^{-1} \partial s) \right) = s(\chi_1 \dot{\gamma}^{-1} \partial s)\, \chi_2 \partial \ddot{\gamma}^{-1} \left(s(\chi_1 \dot{\gamma}^{-1} \partial s) \right) \\
&= s(\chi_1 \dot{\gamma}^{-1} \partial s)\, \chi_2 \left((\dot{\gamma}^{-1} \partial s)(\dot{\gamma}^{-1} \partial \chi_1 \dot{\gamma}^{-1} \partial s) \right) \\
&= s(\chi_1 \dot{\gamma}^{-1} \partial s)\, (\chi_2 \dot{\gamma}^{-1} \partial s)^{\partial \chi_1 \dot{\gamma}^{-1} \partial s} (\chi_2 \dot{\gamma}^{-1} \partial \chi_1 \dot{\gamma}^{-1} \partial s) \\
&= s(\chi_2 \dot{\gamma}^{-1} \partial s)(\chi_1 \dot{\gamma}^{-1} \partial s)(\chi_2 \dot{\gamma}^{-1} \partial \chi_1 \dot{\gamma}^{-1} \partial s) \\
&= s\left((\chi_1 \star_\gamma \chi_2)(\dot{\gamma}^{-1} \partial s) \right) = s^{(\chi_1 \star_\gamma \chi_2)}\,.
\end{aligned}$$

The first crossed module axiom $\eta_{(s^\chi)} = \overline{\chi} \star_\gamma \eta_s \star_\gamma \chi$ is verified by checking

$$\begin{aligned}
(\eta_s \star_\gamma \chi)\, r &= (\chi r)(\ddot{\beta}_\chi \ddot{\gamma}^{-1} \eta_s r) = (\chi r) \left(\ddot{\beta}_\chi \ddot{\gamma}^{-1} ((s^{-1})^{\dot{\gamma} r} s) \right) = (\chi r)(\ddot{\beta}_\chi \ddot{\gamma}^{-1} s^{-1})^{\dot{\beta}_\chi r} (\ddot{\beta}_\chi \ddot{\gamma}^{-1} s) \\
&= \left((s^\chi)^{-1} \right)^{\dot{\gamma} r} (\chi r)(s^\chi) = \left((s^\chi)^{-1} \right)^{\dot{\gamma} r} (s^\chi)(\chi r) \left((s^\chi)^{-1} \right)^{\dot{\gamma}(\dot{\gamma}^{-1} \partial \chi r)} (s^\chi) \\
&= \left(\eta_{(s^\chi)} r \right) (\chi r) \left(\eta_{(s^\chi)} \dot{\gamma}^{-1} \partial \chi r \right) = \left(\chi \star_\gamma \eta_{(s^\chi)} \right) r\,.
\end{aligned}$$

The second crossed module axiom is verified by:

$$s^{\ddot{\imath}s'} \;=\; s^{\eta_{s'}} \;=\; s^{\beta_{\eta_{s'}}} \;=\; \ddot{\beta}_{\eta_{s'}}\ddot{\gamma}^{-1}s \;=\; s(\eta_{s'}\partial\ddot{\gamma}^{-1}s) \;=\; s[s,s'] \;=\; s^{s'}\,.$$

□

[We might show here that (∂, Δ) is a morphism.]

Lemma 3.6

$$(\eta_s)^{\alpha} \;=\; \eta_{(s^{\alpha})}\,.$$

Proof:

$$\begin{aligned}(\eta_s)^{\alpha}q \;&=\; \ddot{\alpha}\ddot{\gamma}^{-1}\eta_s\dot{\alpha}^{-1}\dot{\gamma}q \;=\; \ddot{\alpha}\ddot{\gamma}^{-1}\left((s^{-1})^{\dot{\gamma}\dot{\alpha}^{-1}\dot{\gamma}q}\,s\right) \;=\; \ddot{\alpha}\left((\ddot{\gamma}^{-1}s^{-1})^{\dot{\alpha}^{-1}\dot{\gamma}q}\,(\ddot{\gamma}^{-1}s)\right)\\ &=\; (\ddot{\alpha}\ddot{\gamma}^{-1}s^{-1})^{\dot{\gamma}q}\,(\ddot{\alpha}\ddot{\gamma}^{-1}s) \;=\; ((s^{\alpha})^{-1})^{\dot{\gamma}q}\,(s^{\alpha}) \;=\; \eta_{(s^{\alpha})}q\,.\end{aligned}$$

□

The right-hand square of morphisms of crossed modules in (25) becomes a *crossed square* $\mathcal{S}_{\gamma}(\mathcal{X})$ (see Example 5.9 for the identity case) when the *crossed pairing* (see Section 4) $\boxtimes : R \times W_{\gamma} \to S,\ (r,\chi) \mapsto \chi\dot{\gamma}^{-1}r$, is added to the structure.

3.5 The actor $\mathcal{A}(\mathcal{C})$

(This Subsection (for now) covers only identity derivations and sections.)

The diagram corresponding to equation (25) is

$$\begin{array}{ccccc} G & \xleftarrow{\;\bar{\beta}\;} & G & \xrightarrow{\;\ddot{\kappa}\;} & A(\mathcal{C}) \ltimes W(\mathcal{C}) \\ {\scriptstyle t,h}\downdownarrows\uparrow{\scriptstyle e} & & {\scriptstyle t,h}\downdownarrows\uparrow{\scriptstyle e} & & {\scriptstyle T,H}\downdownarrows\uparrow{\scriptstyle E} \\ R & \xleftarrow[\;\dot{\beta}\;]{} & R & \xrightarrow[\;\dot{\kappa}\;]{} & A(\mathcal{C}) \end{array} \qquad (27)$$

where $W = W(\mathcal{C})$ and $A = A(\mathcal{C})$ are defined as follows:

- W is the group of sections of $\mathcal{C}$ with composition given by equation (18),
- $A = \mathrm{Aut}(\mathcal{C})$ is the group of automorphisms of $\mathcal{C}$,
- $\Delta_{\mathcal{C}} : W \to A,\ \xi \mapsto (\bar{\beta}_{\xi}, \dot{\beta}_{\xi})$ (see equations (19)).

Note: T, H, E were previously written $\Delta_t, \Delta_h, \Delta_e$.

The homomorphisms $\ddot{\kappa}, \dot{\kappa}, T, H, E$ are given as follows.

- $\dot{\kappa} : R \to A,\quad r \mapsto \beta'_r = (\bar{\beta}_r, \dot{\beta}_r) : \mathcal{C} \to \mathcal{C},\quad \bar{\beta}_r g_0 = g_0^r\,,\quad \dot{\beta}_r r_0 = r_0^r = r^{-1}r_0 r\,,$ using the action of R on G in equation (6).
- $\ddot{\kappa} : G \to A \ltimes W,\ g \mapsto (\dot{\kappa}tg, \kappa_g : R \to G)$ where $\kappa_g(r) = (er)^{(etg^{-1})g}$.
- $T(\beta,\xi) = \beta,\quad H(\beta,\xi) = \beta * \Delta_{\mathcal{C}}(\xi),\quad E(\beta) = (\beta, \mathrm{id}).$

[Add in here the associated cat2-group with groups $(A \ltimes W) \ltimes (R \ltimes S),\ A \ltimes W,\ A \ltimes R,\ A$.]

3.6 Actions of a Crossed Module

The material in the rest of this section is taken, in the main, from Norrie's thesis [47]. Recall that an action of a group H on a group G is a group homomorphism from H to the actor of G. The following definition is a straightforward generalisation.

Definition 3.7 *An action of a crossed module* $\mathcal{Y} = (\delta : Q \to P)$ *on a crossed module* $\mathcal{X} = (\partial : S \to R)$ *is a morphism of crossed modules*

$$\alpha = (\ddot{\alpha}, \dot{\alpha}) \quad : \quad \mathcal{Y} \to \mathcal{A}(\mathcal{X}) = \mathrm{Act}(\mathcal{X}) \ ,$$

from $\mathcal{Y}$ *to the* actor *of* $\mathcal{X}$*, as in the following diagram.*

$$\begin{array}{ccccc} S & \xrightarrow{\ddot{\imath}} & W & \xleftarrow{\ddot{\alpha}} & Q \\ \downarrow{\scriptstyle \partial} & & \downarrow{\scriptstyle \Delta} & & \downarrow{\scriptstyle \delta} \\ R & \xrightarrow[\dot{\imath}]{} & A & \xleftarrow[\dot{\alpha}]{} & P \\ & & & & \\ \mathcal{X} & & \mathcal{A}(\mathcal{X}) & & \mathcal{Y} \end{array} \tag{28}$$

Here

- $\mathcal{A}(\mathcal{X})$ is the crossed module $(\Delta : W \to A)$ of Subsection 3.2,
- $\ddot{\alpha}q \ = \ \chi_q : R \to S$, a derivation of $\mathcal{X}$;
- $\dot{\alpha}p \ = \ \beta_p = (\ddot{\beta}_p, \dot{\beta}_p)$, an automorphism of $\mathcal{X}$ giving actions of P on S and R :

$$s^p = \ddot{\beta}_p s \qquad \text{and} \qquad r^p = \dot{\beta}_p r \ .$$

We have seen in Theorem 3.4 that $\iota = (\ddot{\imath}, \dot{\imath})$ is the *inner action* of $\mathcal{X}$ on itself.

Here are five useful identities.

Lemma 3.8

(a) $\partial(s^p) \ = \ \partial\ddot{\beta}_p s \ = \ \dot{\beta}_p \partial s \ = \ (\partial s)^p$;

(b) $s^q \ = \ s^{\delta q} \ = \ \ddot{\beta}_{\delta q} s \ = \ \ddot{\beta}_{\chi_q} s \ = \ s(\chi_q \partial s)$;

(c) $\dot{\beta}_{\delta q} r \ = \ r^{\delta q} \ = \ r^{\dot{\alpha}\delta q} \ = \ r^{\Delta\chi_q} \ = \ \dot{\beta}_{\chi_q} r \ = \ r(\partial\chi_q r)$;

(d) $\chi_{q_1 q_2} r \ = \ (\chi_{q_2} r)(\chi_{q_1} r)^{q_2}$,

(e) $\chi_{q^p} \ = \ \ddot{\beta}_p \circ \chi_q \circ \dot{\beta}_p^{-1}$.

Proof:

(a) $\partial(s^p) \ = \ \partial\ddot{\beta}_p s \ = \ \dot{\beta}_p \partial s \ = \ (\partial s)^p$.

(b) The action of Q on S is via P:

$$s^q \;=\; s^{\delta q} \;=\; \ddot{\beta}_{\delta q} s \;=\; \ddot{\beta}_{\chi_q} s \;=\; s(\chi_q \partial s)\;,$$

and we may check that

$$\begin{aligned}
(s^{q_1})^{q_2} &= (s(\chi_{q_1}\partial s))^{q_2} \\
&= s(\chi_{q_1}\partial s)\; \chi_{q_2}((\partial s)(\partial(\chi_{q_1}\partial s))) \\
&= s(\chi_{q_1}\partial s)(\chi_{q_2}\partial s)^{\partial \chi_{q_1}\partial s}(\chi_{q_2}\partial \chi_{q_1}\partial s) \\
&= s(\chi_{q_2}\partial s)(\chi_{q_1}\partial s)(\chi_{q_2}\partial \chi_{q_1}\partial s) \\
&= s(\chi_{q_1} * \chi_{q_2})(\partial s) \\
&= s^{q_1 q_2}\;.
\end{aligned}$$

(c) $\dot{\beta}_{\delta q} r \;=\; r^{\delta q} \;=\; r^{\dot{\alpha}\delta q} \;=\; r^{\Delta \chi_q} \;=\; \dot{\beta}_{\chi_q} r \;=\; r(\partial \chi_q r)\;.$

(d) $\chi_{q_1 q_2} r \;=\; (\chi_{q_1} \star \chi_{q_2}) r \;=\; (\chi_{q_2} r)(\ddot{\beta}_{\chi_{q_2}} \chi_{q_1} r) \;=\; (\chi_{q_2} r)(\chi_{q_1} r)^{q_2}\;.$

(e) $\chi_{q^p} \;=\; \ddot{\alpha}(q^p) \;=\; (\ddot{\alpha} q)^{\dot{\alpha} p} \;=\; (\chi_q)^{\beta_p} \;=\; \ddot{\beta}_p \circ \chi_q \circ \dot{\beta}_p^{-1}$ by (3.1).

□

[Could do with some more examples of crossed module actions!]

3.7 Semidirect product of crossed modules

Just as a group action gives rise to a semidirect product group (see subsection 1.10), so a crossed module action gives a semidirect product crossed module.

Definition 3.9 The *crossed module semidirect product* with $\mathcal{Y}$ acting on $\mathcal{X}$ is

$$\mathcal{Y} \ltimes \mathcal{X} \;=\; (\,\pi \;:\; Q \ltimes S \longrightarrow P \ltimes R\,)$$

where

(i) $Q \ltimes S$ is the semidirect product with action $s^q \;=\; s^{\delta q} \;=\; \ddot{\beta}_{\delta q} s$ (see Lemma 3.8(b) below);

(ii) $P \ltimes R$ is the semidirect product with action $r^p = \dot{\beta}_p r$;

(iii) $\pi(q,s) \;=\; (\delta q,\, \partial s)$;

(iv) $(q,s)^{(p,r)} \;=\; (q^p,\, (\chi_{q^p}\, r)^{-1}\, s^{pr})$.

Note that the action specified in (iv) is a special case of Proposition 5.14.

Lemma 3.10 *π is a group homomorphism.*

Proof:

$$\begin{aligned}
\pi(\,(q_1,s_1)(q_2,s_2)\,) &= \pi(q_1q_2,\; {s_1}^{q_2}\, s_2) \\
&= (\delta(q_1q_2),\; \partial(\ddot{\beta}_{\delta q_2} s_1)(\partial s_2)) \quad \text{by (i)} \\
\pi(q_1,s_1)\,\pi(q_2,s_2) &= (\delta q_1, \partial s_1)\,(\delta q_2, \partial s_2) \\
&= (\,(\delta q_1)(\delta q_2),\; (\partial s_1)^{\delta q_2}\,(\partial s_2)\,) \\
&= (\delta(q_1q_2),\; (\dot{\beta}_{\delta q_2}\partial s_1)\,(\partial s_2)\,) \quad \text{by (ii)}.
\end{aligned}$$

and the two right-hand sides are equal since $\partial \ddot{\beta}_p = \dot{\beta}_p \partial$ for all $p \in P$. □

Theorem 3.11 *The crossed module $\mathcal{Y} \ltimes \mathcal{X}$ as defined* does *satisfy the two crossed modules axioms.*

Proof: Verification of the first axiom:

$$\begin{aligned}
(p,r)^{-1}\,\pi(q,s)\,(p,r) &= (p,r)^{-1}\,(\delta q, \partial s)\,(p,r) \\
&= (p^{-1},\,(r^{-1})^{p^{-1}})\,((\delta q)p,\,(\partial s)^p\,r\,) \\
&= (p^{-1}(\delta q)p,\,(r^{-1})^{p^{-1}(\delta q)p}\,(\partial s)^p\,r\,) \\
&= (\,\delta(q^p),\,(r^{\delta q^p})^{-1}\,(\partial s^p)\,r\,) \\
&= (\,\delta(q^p),\,(r\,(\partial \chi_{q^p}\,r))^{-1}\,(\partial s^p)\,r\,) && \text{(by 3.8(c))} \\
&= (\,\delta(q^p),\,(\partial \chi_{q^p}\,r)^{-1}\,\partial(s^{pr}\,) \\
&= (\,\delta(q^p),\,\partial((\chi_{q^p}\,r)^{-1}\,s^{pr})\,) \\
&= \pi(\,(q,s)^{(p,r)}\,)\,.
\end{aligned}$$

Verification of the second axiom:

$$\begin{aligned}
(q,s)^{-1}\,(q_1,s_1)\,(q,s) &= (q^{-1},\,((s^{-1})^{q^{-1}})\,(q_1 q,\,s_1{}^q s\,) \\
&= (\,q^{-1}q_1 q,\,(s^{-1})^{q^{-1}q_1 q}\,s_1{}^q\,s\,) \\
&= (\,q_1{}^{\delta q},\,(s^{-1})^{q_1{}^{\delta q}}\,s_1{}^q\,s\,) \\
&= (\,q_1{}^{\delta q},\,s^{-1}\,(\chi_{(q_1{}^{\delta q})}\,(\partial s^{-1}))\,s\,s^{-1}\,s_1{}^{(\delta q)}\,s\,) && \text{(by 3.8(b))} \\
&= (\,q_1{}^{\delta q},\,(\chi_{(q_1{}^{\delta q})}\,(\partial s^{-1}))^{\partial s}\,s_1{}^{(\delta q)(\partial s)}\,) \\
&= (\,q_1{}^{\delta q},\,(\chi_{(q_1{}^{\delta q})}\,\partial s)^{-1}\,s_1{}^{(\delta q)(\partial s)}\,) && \text{(by 2.2(a))} \\
&= (q_1,s_1)^{(\delta q,\,\partial s)} && \text{(by 3.9(iv))} \\
&= (q_1,s_1)^{\pi(q,s)} && \text{(by 3.9(iii))}\,.
\end{aligned}$$

□

3.8 Actions of a Cat1-group

Definition 3.12 *To be added.*

4 Crossed Pairings and Nonabelian Tensor Products

The *nonabelian tensor product* was introduced by Brown and Loday in [17] and developed in Brown, Johnson, Robertson [16].

When G, H are both abelian, $G \otimes H$ is the usual tensor product.

Many computations of the *nonabelain tensor square* $G \otimes G$ of a group G have been made. Here is a small sample of known results:

symmetric group	S_3	C_6
alternating group	A_4	$Q_8 \times C_3$
dihedral groups	D_{2n}, n odd	$\mathbf{Z}_{2n}$
Heisenberg group	$\mathcal{H}$	$\mathbf{Z}^6$

The nonabelian tensor product is a special case of a crossed pairing.

4.1 Compatible Group Actions

Definition 4.1 *Let G and H be groups which act on themselves by conjugation, and also act on each other. These four actions are said to be* compatible *if*

$$ {g_1}^{(h^g)} \;=\; (({g_1}^{g^{-1}})^h)^g \;, \qquad {h_1}^{(g^h)} \;=\; (({h_1}^{h^{-1}})^g)^h \;. $$

Example 4.2 If G, H are normal subgroups of a group Γ, then each acts on the other by conjugation and the actions are compatible.

Example 4.3 Let $\mathcal{X} = (\partial : S \to R)$ be a crossed module. If r^s is defined to be $(\partial s^{-1})r(\partial s)$ then both R and S act on each other and on themselves. Compatibility is easily checked:

- ${s_1}^{(r^s)} \;=\; {s_1}^{(\partial s^{-1})r(\partial s)} \quad {s_1}^{s^{-1}r(\partial s)} \quad {s_1}^{s^{-1}rs}$,
- ${r_1}^{(s^r)} \;=\; (\partial s^r)^{-1} r_1 (\partial s^r) \;=\; (r^{-1}(\partial s)r)^{-1} r_1 (r^{-1}\partial s)r) \;=\; (({r_1}^{r^{-1}})^s)^r$.

4.2 Crossed Pairings

There are two standard definitions of a *crossed pairing*. Here is the one which we shall use.
(There is a more general definition when the two actions are not compatible.)

Definition 4.4 *Let G, H be groups which act compatibly on each other and on a group L.*
A map $\boxtimes : G \times H \to L$, $(g,h) \mapsto g \boxtimes h$, is a crossed pairing if

(a) $\quad (g_1 g_2 \boxtimes h) \;=\; (g_1 \boxtimes h)^{g_2}\, (g_2 \boxtimes h)$,

(b) $\quad (g \boxtimes h_1 h_2) \;=\; (g \boxtimes h_2)\, (g \boxtimes h_1)^{h_2}$,

(c) $\quad (g \boxtimes h)^x \;=\; g^x \boxtimes h^x \quad$ *for all* $x \in G \cup H$.

The alternative definition does not require actions on L and omits axiom (c). It is then observed that $\mathrm{im} \boxtimes \leq L$ inherits G- and H-actions given by (c).

Example 4.5 *If $N \trianglelefteq G$ then a crossed pairing is provided by commutators:*

$$\begin{array}{llll} \text{(a)} & \boxtimes \;:\; G \times N \to N, & g \boxtimes n = [g,n] = (n^{-1})^g\, n, \\ \text{(b)} & \boxtimes \;:\; N \times G \to N, & n \boxtimes g = [n,g] = n^{-1}\, n^g. \end{array}$$

Here are some standard properties of crossed pairings (see Proposition 3 of [16]).

Proposition 4.6 *The following relations hold for all $g, g_1, g_2 \in G$ and for all $h, h_1, h_2 \in H$.*

(a) $(g \boxtimes 1_H) \;=\; (1_G \boxtimes h) \;=\; 1_L$;

(b) $(g \boxtimes h)^{-1} \;=\; (g^{-1} \boxtimes h)^g \;=\; (g \boxtimes h^{-1})^h \;=\; (g^{-1} \boxtimes h^g) \;=\; (g^h \boxtimes h^{-1})$;

(c) $(g_1 \boxtimes h_1)^{h_2 g_2}\, (g_2 \boxtimes h_2) \;=\; (g_2 \boxtimes h_2)\, (g_1 \boxtimes h_1)^{g_2 h_2}$;

(d) $(g^h \boxtimes h_1) \;=\; (g \boxtimes h)^{-1}\, (g \boxtimes h_1)\, (g \boxtimes h)^{h_1}$ *and* $(g_1 \boxtimes h^g) \;=\; (g \boxtimes h)^{g_1}\, (g_1 \boxtimes h)\, (g \boxtimes h)^{-1}$;

(e) $(g_1 g_2 \boxtimes h) \;=\; (g_2 \boxtimes h^{g_1})(g_1 \boxtimes h)$ *and* $(g \boxtimes h_1 h_2) \;=\; (g \boxtimes h_1)(g^{h_1} \boxtimes h_2)$;

(f) $(g \boxtimes h)^{[g_2, h_2]} \;=\; (g_2 \boxtimes h_2)^{-1}\, (g \boxtimes h)\, (g_2 \boxtimes h_2)$;

(g) $(g^{-1}\, g^h \boxtimes h_1) \;=\; (g \boxtimes h)^{-1}\, (g \boxtimes h)^{h_1}$ *and* $(g_1 \boxtimes (h^{-1})^g\, h) \;=\; ((g \boxtimes h)^{-1})^{g_1}\, (g \boxtimes h)$;

(h) $[(g_1 \boxtimes h_1),\, (g_2 \boxtimes h_2)] \;=\; ((g_1^{-1}\, {g_1}^{h_1}) \boxtimes ((h_2^{-1})^{g_2}\, h_2))$.

Proof: Where there are two formulae, the proof of the second mirrors that of the first.

(a) $g \boxtimes h \;=\; g1 \boxtimes h \;=\; (g \boxtimes h)^1\, (1 \boxtimes h)$

(b) $1 \;=\; 1 \boxtimes h \;=\; g^{-1} g \boxtimes h \;=\; (g^{-1} \boxtimes h)^g\, (g \boxtimes h)$

(c)
$$\begin{aligned} g_1 g_2 \boxtimes h_1 h_2 &= (g_1 g_2 \boxtimes h_2)\, (g_1 g_2 \boxtimes h_1)^{h_2} \\ &= (g_1 \boxtimes h_2)^{g_2}\, (g_2 \boxtimes h_2)\, (g_1 \boxtimes h_1)^{g_2 h_2}\, (g_2 \boxtimes h_1)^{h_2} \\ \text{and} \quad g_1 g_2 \boxtimes h_1 h_2 &= (g_1 \boxtimes h_1 h_2)^{g_2}\, (g_2 \boxtimes h_1 h_2) \\ &= (g_1 \boxtimes h_2)^{g_2}\, (g_1 \boxtimes h_1)^{h_2 g_2}\, (g_2 \boxtimes h_2)\, (g_2 \boxtimes h_1)^{h_2} \\ \Rightarrow \quad (g_1 \boxtimes h_1)^{h_2 g_2}\, (g_2 \boxtimes h_2) &= (g_2 \boxtimes h_2)\, (g_1 \boxtimes h_1)^{g_2 h_2} \end{aligned}$$

(d)
$$\begin{aligned} g^h \boxtimes h_1 &= (g \boxtimes {h_1}^{h^{-1}})^h \;=\; (g \boxtimes h h_1 h^{-1})^h \\ &= (g \boxtimes h^{-1})^h\, (g \boxtimes h_1)\, (g \boxtimes h)^{h_1} \\ &= (g \boxtimes h)^{-1}\, (g \boxtimes h_1)\, (g \boxtimes h)^{h_1} \end{aligned}$$

(e) These alternative forms for 4.4 (a),(b) follow immediately from (d).

(f) Substitute $g = {g_1}^{h_2 g_2},\; h = {h_1}^{h_2 g_2}$ in (c).

(g)
$$\begin{aligned} (g^{-1} g^h) \boxtimes h_1 &= (g^{-1} \boxtimes h_1)^{g^h}\, (g^h \boxtimes h_1) \;=\; (g^{-1} \boxtimes h_1^g)^{[g,h]}\, (g^h \boxtimes h_1) \\ &= (g \boxtimes h)^{-1}\, (g \boxtimes h_1)^{-1}\, (g \boxtimes h).(g \boxtimes h)^{-1}\, (g \boxtimes h_1)\, (g \boxtimes h)^{h_1} \quad \text{by (b),(d),(e)} \\ &= (g \boxtimes h)^{-1}\, (g \boxtimes h)^{h_1} \end{aligned}$$

(h)
$$\begin{aligned}[g_1 \boxtimes h_1, g_2 \boxtimes h_2] &= (g_1 \boxtimes h_1)^{-1}\,(g_2 \boxtimes h_2)^{-1}\,(g_1 \boxtimes h_1)\,(g_2 \boxtimes h_2) \\ &= (g_1 \boxtimes h_1)^{-1}\,(g_1 \boxtimes h_1)^{[g_2,h_2]} \;=\; (g_1 \boxtimes h_1)^{-1}\,(g_1 \boxtimes h_1)^{(h_2^{-1})^{g_2} h_2} \quad \text{by (e)} \\ &= ((g_1^{-1}\,{g_1}^{h_1}) \boxtimes ((h_2^{-1})^{g_2}\,h_2)) \quad \text{by (f).}\end{aligned}$$

□

Lemma 4.7 *The* principal crossed pairing *of a crossed module $\mathcal{X} = (\partial : S \to R)$ is given by*

$$\boxtimes : R \times S \to S,\ (r,s) \mapsto \eta_s(r) = (s^{-1})^r s\ .$$

Proof: We have seen in Example 4.3 that R and S have compatible actions. The three axioms are easily checked.

(a)
$$(r_1 \boxtimes s)^{r_2}(r_2 \boxtimes s) \;=\; ((s^{-1})^{r_1} s)^{r_2}((s^{-1})^{r_2} s) \;=\; (s^{-1})^{r_1 r_2} s \;=\; r_1 r_2 \boxtimes s$$

(b)
$$(r \boxtimes s_1)(r \boxtimes s_1)^{s_2} \;=\; ((s_2^{-1})^r s_2)(s_2^{-1}(s_1^{-1})^r s_1 s_2) \;=\; ((s_1 s_2)^{-1})^r (s_1 s_2) \;=\; r \boxtimes s_1 s_2$$

(c)
$$\begin{aligned} r^{r_0} \boxtimes s^{r_0} &= ((s^{r_0})^{-1})^{r^{r_0}} s^{r_0} \;=\; (s^{-1})^{r r_0} s^{r_0} \;=\; (r \boxtimes s)^{r_0} \\ r^{\partial s_0} \boxtimes s^{s_0} &= ((s^{s_0})^{-1})^{s_0^{-1} r s_0} s^{s_0} \;=\; ((s^{-1})^r s)^{s_0} \;=\; (r \boxtimes s)^{s_0} \end{aligned}$$

□

So we may write principal derivations as $\eta_s r = r \boxtimes s$ and principal sections as $\kappa_s r = (r, r \boxtimes s)$.

A standard result concerning crossed pairings shows that the nonabelian tensor product is the universal object for this construction.

Definition 4.8 *Given groups G and H which act compatibly on each other, the* nonabelian tensor product $G \otimes H$ *of G and H has generating set $\{g \otimes h \mid g \in G,\ h \in H\}$ subject to relations*

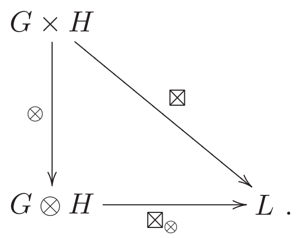
$$(g_1 g_2 \otimes h) \;=\; (g_1 \otimes h)^{g_2}\,(g_2 \otimes h)\ , \quad (g \otimes h_1 h_2) \;=\; (g \otimes h_2)\,(g \otimes h_1)^{h_2}\ , \tag{29}$$

where
$$(g \otimes h)^x \;=\; (g^x \otimes h^x) \quad \text{for all } x \in G \cup H\ .$$

Theorem 4.9 *The nonabelian tensor product function*

$$\otimes \;:\; G \times H \to G \otimes H,\ \ (g,h) \mapsto g \otimes h$$

is a crossed pairing. Moreover, given any crossed pairing $\boxtimes : G \times H \to L$, there is a unique homomorphism $\boxtimes_\otimes : G \otimes H \to L$ satisfying $\boxtimes = \boxtimes_\otimes \circ \otimes$ so that the following diagram commutes:



Checking that a potential crossed pairing satisfies the axioms of Definition 4.4 can be a tedious process. However we can convert this into checking that maps to certain semidirect products are homomorphisms.

Lemma 4.10 *Let $\boxtimes : G \times H \to L$ be a crossed pairing. Then*

(a) *given a fixed element $h \in H$, the map*

$$\theta_h \ : \ G \to G \ltimes L, \ \ g \mapsto (g,\, g \boxtimes h)$$

is a group homomorphism;

(b) *given a fixed element $g \in G$, the map*

$$\theta_g \ : \ H \to H \ltimes L, \ \ h \mapsto (h, (g \boxtimes h)^{-1})$$

is a group homomorphism.

Proof:

$$\theta_h(g_1 g_2) \ = \ (g_1 g_2,\, (g_1 \boxtimes h)^{g_2}\, (g_2 \boxtimes h)) \ = \ (g_1,\, g_1 \boxtimes h)\, (g_2,\, g_2 \boxtimes h) \ = \ (\theta_h g_1)\, (\theta_h g_2)\ .$$

$$\theta_g(h_1 h_2) \ = \ (h_1 h_2,\ \{(g \boxtimes h_2)\, ((g \boxtimes h_1))^{h_2}\}^{-1}) \ = \ (h_1,\, (g \boxtimes h_1)^{-1})\, (h_2,\, (g \boxtimes h_2)^{-1}) \ = \ (\theta_g h_1)\, (\theta_g h_2)\ .$$

□

The converse proposition gives a way of checking that a given map is a crossed pairing.

Proposition 4.11 *Given a map $\odot : G \times H \to L$ and an action $(g \odot h)^x = g^x \odot h^x$ for all $x \in G \cup H$, such that for all $h \in H$ and $g \in G$*

- *$\theta_h : G \to G \ltimes L,\ g \mapsto (g, g \odot h)$ is a homomorphism, and*
- *$\theta_g : H \to H \ltimes L,\ h \mapsto (h, (g \odot h)^{-1})$ is a homomorphism,*

then $\odot$ is a crossed pairing.

Proof:

$$\begin{aligned} (g_1 g_2,\, g_1 g_2 \odot h) \ &= \ \theta_h(g_1 g_2) \ = \ (\theta_h g_1)(\theta_h g_2) \\ &= \ (g_1,\, g_1 \odot h)(g_2,\, g_2 \odot h) \\ &= \ (g_1 g_2,\, (g_1 \odot h)^{g_2}\, (g_2 \odot h))\ . \end{aligned}$$

A similar argument shows that (4.4)(b) is also satisfied, so $\odot$ is a crossed pairing. □

5 Crossed Squares

Crossed squares were introduced by Guin-Waléry and Loday (see, for example, [36, 41, 17]) as fundamental crossed squares of commutative squares of spaces, but are also of purely algebraic interest. We denote by $[n]$ the set $\{1, 2, \ldots, n\}$. We use the $n = 2$ version of the definition of crossed n-cube as given by Ellis and Steiner [30].

Definition 5.1 *A crossed square consists of the following:*

- *a commutative diagram of group homomorphisms*

$$\mathcal{R} \quad = \quad \begin{array}{ccc} R_{[2]} & \xrightarrow{\ \ddot{\partial}_1\ } & R_{\{2\}} \\ {\scriptstyle \ddot{\partial}_2}\downarrow & & \downarrow{\scriptstyle \dot{\partial}_2} \\ R_{\{1\}} & \xrightarrow[\ \dot{\partial}_1\ ]{} & R_{\emptyset} \end{array} \quad ; \tag{30}$$

- *actions of $R_\emptyset$ on $R_{\{1\}}, R_{\{2\}}$ and $R_{[2]}$ which determine actions of $R_{\{1\}}$ on $R_{\{2\}}$ and $R_{[2]}$ via $\dot{\partial}_1$ and actions of $R_{\{2\}}$ on $R_{\{1\}}$ and $R_{[2]}$ via $\dot{\partial}_2$;*

- *a function $\boxtimes : R_{\{1\}} \times R_{\{2\}} \to R_{[2]}$.*

The following axioms must be satisfied for all $l \in R_{[2]},\ m, m_1, m_2 \in R_{\{1\}},\ n, n_1, n_2 \in R_{\{2\}},\ p \in R_\emptyset$:

(a) *the homomorphisms $\ddot{\partial}_1, \ddot{\partial}_2$ preserve the action of $R_\emptyset$;*

(b) *each of $\ddot{\mathcal{R}}_1 = (\ddot{\partial}_1 : R_{[2]} \to R_{\{2\}}),\ \ddot{\mathcal{R}}_2 = (\ddot{\partial}_2 : R_{[2]} \to R_{\{1\}}),\ \dot{\mathcal{R}}_1 = (\dot{\partial}_1 : R_{\{1\}} \to R_\emptyset),\ \dot{\mathcal{R}}_2 = (\dot{\partial}_2 : R_{\{2\}} \to R_\emptyset)$ and the diagonal $\mathcal{R}_{[2]} = (\partial_{[2]} = \dot{\partial}_1\ddot{\partial}_2 = \dot{\partial}_2\ddot{\partial}_1 : R_{[2]} \to R_\emptyset)$ are crossed modules (with actions via $R_\emptyset$);*

(c) $\boxtimes$ *is a* crossed pairing*:*

$$\begin{aligned} &\text{(i)}\ \ (m_1 m_2 \boxtimes n) \ = \ (m_1 \boxtimes n)^{m_2}\,(m_2 \boxtimes n)\ , \\ &\text{(ii)}\ \ (m \boxtimes n_1 n_2) \ = \ (m \boxtimes n_2)\,(m \boxtimes n_1)^{n_2}\ , \\ &\text{(iii)}\ \ (m \boxtimes n)^p \ = \ (m^p \boxtimes n^p)\ ; \end{aligned}$$

(d) $\quad \ddot{\partial}_1(m \boxtimes n) \ = \ (n^{-1})^m\, n \quad$ *and* $\quad \ddot{\partial}_2(m \boxtimes n) \ = \ m^{-1}\, m^n$,

(e) $\quad (m \boxtimes \ddot{\partial}_1 l) \ = \ (l^{-1})^m\, l \quad$ *and* $\quad (\ddot{\partial}_2 l \boxtimes n) \ = \ l^{-1}\, l^n$.

Note that the actions of $R_{\{1\}}$ on $R_{\{2\}}$ and $R_{\{2\}}$ on $R_{\{1\}}$ via $R_\emptyset$ are compatible since

$$m_1{}^{(n^m)} \ = \ m_1{}^{\dot{\partial}_2(n^m)} \ = \ m_1{}^{m^{-1}(\dot{\partial}_2 n)m} \ = \ ((m_1{}^{m^{-1}})^n)^m\ .$$

Lemma 5.2 *Pairs $\partial_1 = (\ddot{\partial}_1, \dot{\partial}_1)$ and $\partial_2 = (\ddot{\partial}_2, \dot{\partial}_2)$ are morphisms of crossed modules, and so also are $(1_{R_{[2]}}, \dot{\partial}_1)$, $(1_{R_{[2]}}, \dot{\partial}_2)$, $(\ddot{\partial}_2, 1_{R_\emptyset})$ and $(\ddot{\partial}_1, 1_{R_\emptyset})$.*

Proof: For ∂_1 we note that (30) commutes, that $\ddot{\partial}_1(\ell^m) = (\ddot{\partial}_1 l)^m$ by (a), and that $(\ddot{\partial}_1\ell)^m = (\ddot{\partial}_1\ell)^{\dot{\partial}_1 m}$ by (b). The arguments for the other five morphisms are similar. $\square$

Note in particular that

$$\ddot{\partial}_1(\ell^p) = (\ddot{\partial}_1\ell)^p \qquad \text{and} \qquad \ddot{\partial}_2(\ell^p) = (\ddot{\partial}_2\ell)^p.$$

Lemma 5.3 *In the crossed square $\mathcal{R}$ above:*

(a)

$$\ell^{(m\boxtimes n)} \;=\; \ell^{[m,n]}\,,$$

(b)

$$\ell^{nm}\,(m\boxtimes n) \;=\; (m\boxtimes n)\,\ell^{mn} \qquad \textit{and} \qquad \ell^{(n^m)}\,(m\boxtimes n) \;=\; (m\boxtimes n)\,\ell^{n}\,.$$

(c)

$$m^{\ddot{\partial}_1\ell} = m^{\ddot{\partial}_2\ell} \qquad \textit{and} \qquad n^{\ddot{\partial}_2\ell} = n^{\ddot{\partial}_1\ell}\,.$$

Proof:

(a)

$$\ell^{(m\boxtimes n)} \;=\; \ell^{\dot{\partial}_1\ddot{\partial}_2(m\boxtimes n)} \;=\; \ell^{m^{-1}\,m^n} \;=\; \ell^{m^{-1}n^{-1}mn} \;=\; \ell^{[m,n]}\,.$$

(b) The first identity is given by

$$(m\boxtimes n)^{-1}\,\ell^{nm}\,(m\boxtimes n) \;=\; \ell^{nm\ddot{\partial}_1(m\boxtimes n)} \;=\; \ell^{nm(n^{-1m})n} \;=\; \ell^{mn}\,.$$

Then replace ℓ by $\ell^{m^{-1}}$ to get the second.

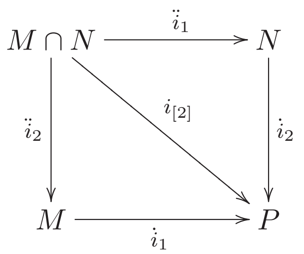
(c)

$$m^{\ddot{\partial}_1\ell} \;=\; m^{\dot{\partial}_2\ddot{\partial}_1\ell} \;=\; m^{\dot{\partial}_1\ddot{\partial}_2\ell} \;=\; m^{\ddot{\partial}_2\ell}$$

$\square$

5.1 Examples of crossed squares

Example 5.4 If M, N are normal subgroups of the group P then the diagram of inclusions



together with the actions of P on M, N and $M\cap N$ given by conjugation and the function

$$\boxtimes \;:\; M\times N \to M\cap N, \quad (m,n)\mapsto [m,n] \;=\; m^{-1}n^{-1}mn$$

is a crossed square. We may check the axioms as follows:

(a) *The identity maps preserve P-actions.*

(b) *The five crossed modules are all conjugation crossed modules.*

(c) *We verify the three commutator identities:*

(i)

$$\begin{aligned}(m_1m_2 \boxtimes n) \;=\; [m_1m_2, n] &= (m_1m_2)^{-1}n^{-1}(m_1m_2)n \\ &= m_2^{-1}\{m_1^{-1}n^{-1}m_1n\}m_2\, m_2^{-1}n^{-1}m_2n \\ &= [m_1, n]^{m_2}\, [m_2, n] \ ;\end{aligned}$$

(ii)

$$\begin{aligned}(m \boxtimes n_1n_2) \;=\; [m, n_1n_2] &= m^{-1}(n_1n_2)^{-1}m(n_1n_2) \\ &= m^{-1}n_2^{-1}mn_2\, n_2^{-1}\{m^{-1}n_1^{-1}mn_1\}n_2 \\ &= [m, n_2]\, [m, n_1]^{n_2} \ ;\end{aligned}$$

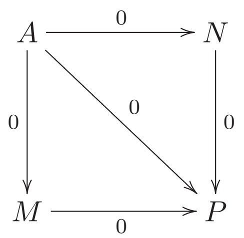
(iii)

$$\begin{aligned}(m^p \boxtimes n^p) \;=\; [m^p, n^p] &= \{p^{-1}mp\}^{-1}\, \{p^{-1}np\}^{-1}\, \{p^{-1}mp\}\{p^{-1}np\} \\ &= p^{-1}m^{-1}p\, p^{-1}n^{-1}p\, p^{-1}mp\, p^{-1}np \\ &= p^{-1}[m, n]p \;=\; [m, n]^p \ .\end{aligned}$$

(d) $\ddot{i}_1(m \boxtimes n) \;=\; (m^{-1}n^{-1}m)n \;=\; n^{-1^m}\, n$ and $\ddot{i}_2(m \boxtimes n) \;=\; m^{-1}\,(n^{-1}mn) \;=\; m^{-1}\, m^n$.

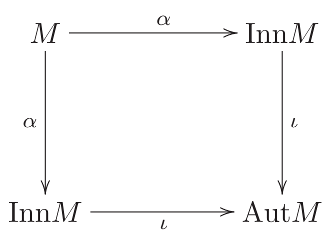
(e) $(m \boxtimes \ddot{i}_1 l) \;=\; (m^{-1}l^{-1}m)l \;=\; l^{-1^m}\, l$ and $(\ddot{i}_2 l \boxtimes n) \;=\; l^{-1}\,(n^{-1}ln) \;=\; l^{-1}\, l^n$.

Example 5.5 If M, N are ordinary $P-$modules and A is an arbitrary abelian group on which P is assumed to act trivially, then there is a crossed square



Note that M acts trivially on N, and conversely, and that $m \boxtimes n = 1_A$.

Example 5.6 The diagram



is a crossed square, where α maps $m \in M$ to the inner automorphism $\beta_m : M \to M,\ m' \mapsto m^{-1}m'm$; where ι is the inclusion of Inn$M$ in Aut$M$; the actions are standard; and the crossed pairing is

$$\boxtimes \ :\ \mathrm{Inn}M \times \mathrm{Inn}M \to M, \quad (\beta_m, \beta_{m'}) \ \mapsto\ [m, m'] \ .$$

Example 5.7 If U, V are subspaces of a space X with a point x_0 in common, then the diagram of boundary maps

$$\begin{array}{ccc} \pi_3(X; U, V, x_0) & \longrightarrow & \pi_2(V, U \cap V, x_0) \\ \downarrow & & \downarrow \\ \pi_2(U, U \cap V, x_0) & \longrightarrow & \pi_1(U \cap V, x_0) \end{array}$$

in which $\pi_3(X; U, V, x_0)$ is the triad homotopy group, together with the standard actions and the triad Whitehead product

$$\boxtimes \; : \; \pi_2(U, U \cap V, x_0) \times \pi_2(V, U \cap V, x_0) \; \to \; \pi_3(X; U, V, x_0)$$

is a crossed square.

Lemma 5.8 *The* transpose

$$\tilde{\mathcal{R}} \quad = \quad \begin{array}{ccc} R_{[2]} & \xrightarrow{\ddot{\partial}_2} & R_{\{1\}} \\ {\scriptstyle \ddot{\partial}_1}\downarrow & & \downarrow{\scriptstyle \dot{\partial}_1} \\ R_{\{2\}} & \xrightarrow[\dot{\partial}_2]{} & R_{\emptyset} \end{array} \qquad \textit{of} \qquad \mathcal{R} \quad = \quad \begin{array}{ccc} R_{[2]} & \xrightarrow{\ddot{\partial}_1} & R_{\{2\}} \\ {\scriptstyle \ddot{\partial}_2}\downarrow & & \downarrow{\scriptstyle \dot{\partial}_2} \\ R_{\{1\}} & \xrightarrow[\dot{\partial}_1]{} & R_{\emptyset} \end{array}$$

is a crossed square with crossed pairing

$$\tilde{\boxtimes} \;\; : \; R_{\{2\}} \times R_{\{1\}} \to R_{[2]}, \quad (n, m) \; \mapsto \; n \,\tilde{\boxtimes}\, m := (m \boxtimes n)^{-1} \; . \tag{31}$$

Proof:

$$\begin{array}{rcl} n_1 n_2 \,\tilde{\boxtimes}\, m & = & (m \boxtimes n_1 n_2)^{-1} \; = \; ((m \boxtimes n_2)\,(m \boxtimes n_1)^{n_2})^{-1} \\ & = & ((m \boxtimes n_1)^{n_2})^{-1}\,(m \boxtimes n_2)^{-1} \; = \; (n_1 \,\tilde{\boxtimes}\, m)^{n_2}\,(n_2 \,\tilde{\boxtimes}\, m) \; ; \\ n \,\tilde{\boxtimes}\, m_1 m_2 & = & (m_1 m_2 \boxtimes n)^{-1} \; = \; ((m_1 \boxtimes n)^{m_2}\,(m_2 \boxtimes n))^{-1} \\ & = & ((m_2 \boxtimes n)^{-1}\,((m_1 \boxtimes n)^{m_2})^{-1} \; = \; (n \,\tilde{\boxtimes}\, m_2)\,(n \,\tilde{\boxtimes}\, m_1)^{m_2} \; . \end{array}$$

$\square$

Example 5.9 The actor $\mathcal{A}(\mathcal{X})$ of a crossed module $\mathcal{X}$ (see subsection 3.2)

$$\mathcal{A} \quad = \quad \begin{array}{ccc} S & \xrightarrow{\ddot{\imath}} & W \\ {\scriptstyle \partial}\downarrow & & \downarrow{\scriptstyle \Delta} \\ R & \xrightarrow[\imath]{} & A \end{array} \tag{32}$$

is a crossed square with crossed pairing

$$\boxtimes \;:\; R \times W \;\to\; S, \quad (r,\chi) \;\mapsto\; \chi r \;.$$

We already know that the square $\mathcal{A}$ contains 5 crossed modules, but we still need to check the axioms which involve the crossed pairing:

(c) (i) $(qr \boxtimes \chi) \;=\; \chi(qr) \;=\; (\chi q)^r\,(\chi r) \;=\; (q \boxtimes \chi)^r\,(r \boxtimes \chi)$.

(ii) $(r \boxtimes \chi_1 \star \chi_2) \;=\; (\chi_2 r)(\chi_1 r)(\chi_2 \partial \chi_1 r) \;=\; (\chi_2 r)\,(\chi_1 r)^{\chi_2} \;=\; (r \boxtimes \chi_2)\,(r \boxtimes \chi_1)^{\chi_2}$,

using the action in Lemma 3.5.

(iii) $(r^\beta \boxtimes \chi^\beta) \;=\; \chi^\beta(r^\beta) \;=\; (\ddot{\beta}\chi\dot{\beta}^{-1})(\dot{\beta}r) \;=\; \ddot{\beta}(\chi r) \;=\; (r \boxtimes \chi)^\beta$.

(d) Since $\ddot{\imath}(r \boxtimes \chi) = \ddot{\imath}(\chi r) = \eta_{\chi r}$, we wish to prove that

$$\eta_{\chi r} = (\chi^{-1})^r \star \chi \quad \text{or, equivalently,} \quad \chi^r = \chi \star (\eta_{\chi r})^{-1} = \chi \star \eta_{(\chi r)^{-1}}\,.$$

Starting with the right-hand side,

$$\begin{array}{rcll} (\chi \star \eta_{(\chi r)^{-1}})q & = & (\eta_{(\chi r)^{-1}}q)(\ddot{\beta}_{\eta_{(\chi r)^{-1}}}\chi q) & \text{by Lemma 2.4 (c)} \\ & = & (\chi r)^q (\chi r)^{-1} (\chi q)^{(\chi r)^{-1}} & \text{by Lemma 2.8 (b)} \\ & = & (\chi r)^q (\chi q)(\chi r)^{-1} & \\ & = & (\chi r)^q (\chi q)(\chi r^{-1})^r & \text{by Lemma 2.2 (b)} \\ & = & \ddot{\beta}_r((\chi r)^{qr^{-1}} (\chi q)^{r^{-1}} (\chi r^{-1})) & \\ & = & \ddot{\beta}_r \chi(rqr^{-1}) \;=\; (\ddot{\beta}_r \chi \dot{\beta}_r^{-1})q & \text{by Lemma 2.4 (c)} \\ & = & \chi^{ir} q \;=\; \chi^r q\,. & \end{array}$$

The second formula follows by
$\partial(r \boxtimes \chi) \;=\; \partial\chi(r) \;=\; r^{-1}r(\partial\chi r) \;=\; r^{-1}(\dot{\beta}_\chi r) \;=\; r^{-1}r^\chi$.

(e) $(r \boxtimes \ddot{\imath}(s)) \;=\; \eta_s(r) \;=\; (s^{-1})^r s$ by Lemma 2.7,
$(\partial s \boxtimes \chi) \;=\; \chi(\partial s) \;=\; s^{-1}s(\chi \partial s) \;=\; s^{-1}(\ddot{\beta}_\chi s) \;=\; s^{-1}s^\chi$,

5.2 Morphisms of crossed squares

A morphism $\theta : \mathcal{R} \to \mathcal{S}$ of crossed squares is a 4-tuple of group homomorphisms which commute with the morphisms in $\mathcal{R}$ and $\mathcal{S}$ and preserve all the actions and the crossed pairings.

Definition 5.10 *A morphism $\theta : \mathcal{R} \to \mathcal{S}$ of crossed squares consists of four group homomorphisms*

$$\theta_{[2]} : R_{[2]} \to S_{[2]}, \quad \theta_{\{2\}} : R_{\{2\}} \to S_{\{2\}}, \quad \theta_{\{1\}} : R_{\{1\}} \to S_{\{1\}}, \quad \theta_{\emptyset} : R_{\emptyset} \to S_{\emptyset},$$

forming a commutative cube with the morphisms $\ddot{\partial}_j, \dot{\partial}_j$ of $\mathcal{R}$ and $\ddot{\gamma}_j, \dot{\gamma}_j$ of $\mathcal{S}$ (for $j \in \{1,2\}$), which pair off in appropriate ways to form crossed module morphisms

$$(\theta_{[2]}, \theta_{\{2\}}) : \ddot{\mathcal{R}}_1 \to \ddot{\mathcal{S}}_1, \quad (\theta_{[2]}, \theta_{\{1\}}) : \ddot{\mathcal{R}}_2 \to \ddot{\mathcal{S}}_2, \quad (\theta_{\{2\}}, \theta_{\emptyset}) : \dot{\mathcal{R}}_2 \to \dot{\mathcal{S}}_2, \quad (\theta_{\{1\}}, \theta_{\emptyset}) : \dot{\mathcal{R}}_1 \to \dot{\mathcal{S}}_1,$$

and which preserve the crossed pairing:

$$\theta_{[2]}(m \boxtimes_{\mathcal{R}} n) \;=\; (\theta_{\{1\}} m) \boxtimes_{\mathcal{S}} (\theta_{\{2\}} n).$$

Definition 5.11 *The group* $\mathrm{Aut}(\mathcal{R})$ *of automorphisms of the crossed square $\mathcal{R}$ is*

$$\mathrm{Aut}(\mathcal{R}) \;=\; \{\alpha = (\alpha_{[2]}, \alpha_{\{1\}}, \alpha_{\{2\}}, \alpha_{\emptyset}) \;:\; \mathcal{R} \to \mathcal{R}\}$$

such that $(\alpha_{[2]}, \alpha_{\{1\}})$ is an automorphism of $\ddot{\mathcal{R}}_2$, $(\alpha_{[2]}, \alpha_{\{2\}})$ is an automorphism of $\ddot{\mathcal{R}}_1$, $(\alpha_{\{2\}}, \alpha_{\emptyset})$ is an automorphism of $\dot{\mathcal{R}}_2$, and $(\alpha_{\{1\}}, \alpha_{\emptyset})$ is an automorphism of $\dot{\mathcal{R}}_1$.

[Do we also require the following?]

$$\alpha_{[2]}(m \boxtimes n) \;=\; (\alpha_{\{1\}} m) \boxtimes (\alpha_{\{2\}} n) \;\; ?$$

[Is there a crossed square version of Theorem 1.8 ?]

Theorem 5.12 *Every crossed square is a quotient of normal inclusion crossed squares* **?**

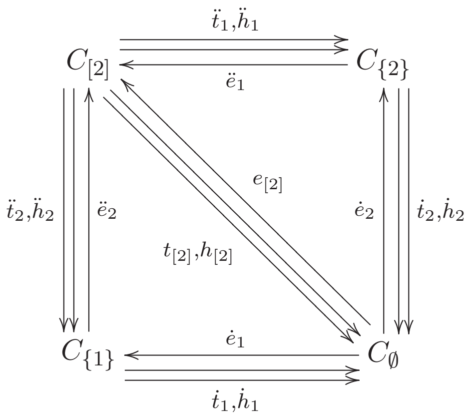
(Note: notes 10/7/03 only go one way.)

5.3 Cat^2-groups

We take the definition of a cat^2-group from Section 5 of Brown and Loday [17], suitably modified.

Definition 5.13 A cat^2-group $\mathcal{C} = (C_{[2]}, C_{\{2\}}, C_{\{1\}}, C_{\emptyset})$ comprises 4 groups and 15 homomorphisms, as shown in the following diagram,

$\mathcal{C} \quad =$



(33)

subject the following axioms:

- the four sides of the square are cat^1-groups, denoted $\ddot{\mathcal{C}}_1, \ddot{\mathcal{C}}_2, \dot{\mathcal{C}}_1, \dot{\mathcal{C}}_2$,
- $\dot{t}_1\ddot{h}_2 = \dot{h}_2\ddot{t}_1, \quad \dot{t}_2\ddot{h}_1 = \dot{h}_1\ddot{t}_2, \quad \dot{e}_1\dot{t}_2 = \ddot{t}_2\ddot{e}_1, \quad \dot{e}_2\dot{t}_1 = \ddot{t}_1\ddot{e}_2, \quad \dot{e}_1\dot{h}_2 = \ddot{h}_2\ddot{e}_1, \quad \dot{e}_2\dot{h}_1 = \ddot{h}_1\ddot{e}_2,$
- $\dot{t}_1\ddot{t}_2 = \dot{t}_2\ddot{t}_1 = t_{[2]}, \quad \dot{h}_1\ddot{h}_2 = \dot{h}_2\ddot{h}_1 = h_{[2]}, \quad \dot{e}_1\ddot{e}_2 = \dot{e}_2\ddot{e}_1 = e_{[2]},$
  making the diagonal a pre-cat^1-group $(e_{[2]}; t_{[2]}, h_{[2]} : C_{[2]} \to C_{\emptyset})$.

It follows from these identities that $(\ddot{t}_1, \dot{t}_1)$, $(\ddot{h}_1, \dot{h}_1)$ and $(\ddot{e}_1, \dot{e}_1)$ are morphisms of cat^1-groups.

[Add here the Peiffer subgroup of the diagonal pre-cat^1-group.]

5.4 The cat^2-group associated to a crossed square

Given a crossed square

$$\mathcal{R} \quad = \quad \begin{array}{ccc} R_{[2]} & \xrightarrow{\ddot{\partial}_1} & R_{\{2\}} \\ {\scriptstyle \ddot{\partial}_2}\downarrow & & \downarrow{\scriptstyle \dot{\partial}_2} \\ R_{\{1\}} & \xrightarrow[\dot{\partial}_1]{} & R_{\emptyset} \end{array} \tag{34}$$

with crossed pairing $\boxtimes : R_{\{1\}} \times R_{\{2\}} \to R_{[2]}$, we wish to construct an associated cat^2-group.

Proposition 5.14 *For $\mathcal{R}$ a crossed square (as in Definition 5.1) there are group actions of $R_{\emptyset} \ltimes R_{\{2\}}$ on $R_{\{1\}} \ltimes R_{[2]}$ and $R_{\emptyset} \ltimes R_{\{1\}}$ on $R_{\{2\}} \ltimes R_{[2]}$ given by*

$$(m, \ell)^{(p,n)} = (m^p, (m^p \boxtimes n)\, \ell^{pn}) \ , \tag{35}$$
$$(n, \ell)^{(p,m)} = (n^p, (m \boxtimes n^p)^{-1}\, \ell^{pm}) \ . \tag{36}$$

Proof: There are two axioms to be checked for the first identity:

$$\begin{aligned} (m_1, \ell_1)^{(p,n)}(m_2, \ell_2)^{(p,n)} &= (m_1^p, (m_1^p \boxtimes n)\ell_1^{pn})(m_2^p, (m_2^p \boxtimes n)\ell_2^{pn}) \\ &= (m_1^p m_2^p, (m_1^p \boxtimes n)^{m_2^p}[\ell_1^{pnm_2^p}(m_2^p \boxtimes n)]\ell_2^{pn}) \\ &= (m_1^p m_2^p, (m_1^p \boxtimes n)^{m_2^p}[(m_2^p \boxtimes n)\ell_1^{m_2 pn}]\ell_2^{pn}) \qquad \text{by Lemma 5.3(b)} \\ &= ((m_1 m_2)^p, ((m_1 m_2)^p \boxtimes n)(\ell_1^{m_2}\ell_2)^{pn}) \\ &= (m_1 m_2, \ell_1^{m_2}\ell_2)^{(p,n)} \\ &= ((m_1, \ell_1)(m_2, \ell_2))^{(p,n)} \ , \end{aligned}$$

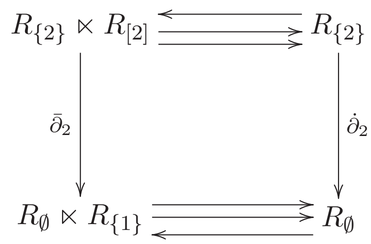
$$\begin{aligned} ((m, \ell)^{(p_1,n_1)})^{(p_2,n_2)} &= (m^{p_1}, (m^{p_1} \boxtimes n_1)\ell^{p_1 n_1})^{(p_2,n_2)} \\ &= ((m^{p_1})^{p_2}, (m^{p_1 p_2} \boxtimes n_2)((m^{p_1} \boxtimes n_1)\ell^{p_1 n_1})^{p_2 n_2}) \\ &= (m^{p_1 p_2}, (m^{p_1 p_2} \boxtimes n_2)(m^{p_1 p_2} \boxtimes {n_1}^{p_2})^{n_2}\ell^{p_1 n_1 p_2 n_2}) \\ &= (m^{p_1 p_2}, (m^{p_1 p_2} \boxtimes {n_1}^{p_2} n_2)\ell^{p_1 p_2 {n_1}^{p_2} n_2}) \\ &= (m, \ell)^{(p_1 p_2, {n_1}^{p_2} n_2)} \\ &= (m, \ell)^{((p_1,n_1)(p_2,n_2))} \ . \end{aligned}$$

The second identity follows using the transpose crossed pairing (31). □

In [2] we noted that the cat^1-group associated to a crossed module $\mathcal{X}$ has homomorphisms

$$t, h : R \ltimes S \to S, \quad t(r, s) = r, \quad h(r, s) = r(\partial s) \ .$$

Applying this construction to $\ddot{\mathcal{R}}_1$ and $\dot{\mathcal{R}}_1$ we obtain a *crossed module of cat^1-groups*:



(37)

Lemma 5.15 *The action given in Proposition 5.14 makes* $(\bar\partial_2 : R_{\{2\}} \ltimes R_{[2]} \to R_\emptyset \ltimes R_{\{1\}})$ *a crossed module.*

Proof: X1:

$$\begin{aligned}
\bar\partial_2((n,\ell)^{(p,m)}) &= \bar\partial_2(n^p, (m \boxtimes n^p)^{-1}\ell^{pm}) \\
&= (\dot\partial_2(n^p), \ddot\partial_2(m \boxtimes n^p)^{-1}\ddot\partial_2(\ell^{pm})) \\
&= (\dot\partial_2(n^p), (m^{-1}m^{\dot\partial_2 n^p})^{-1}\ddot\partial_2(\ell^{pm})) \\
&= (\dot\partial_2(n^p), (m^{-1})^{(\dot\partial_2 n^p)} m \ddot\partial_2(\ell^{pm})) \\
&= (\dot\partial_2(n^p), (m^{-1})^{\dot\partial_2(n^p)}(\ddot\partial_2\ell^p)m) \\
&= (p^{-1}(\dot\partial_2 n)p, (m^{-1})^{p^{-1}(\dot\partial_2 n)p}\ddot\partial_2(\ell^p)m) \\
&= (p^{-1}, (m^{-1})^{p^{-1}})(\dot\partial_2 n, \ddot\partial_2\ell)(p,m) \\
&= (p,m)^{-1}\ \bar\partial_2(n,\ell)(p,m)\ .
\end{aligned}$$

X2:

$$\begin{aligned}
(n_1,\ell_1)^{\bar\partial_2(n_2,\ell_2)} &= (n_1,\ell_1)^{(\dot\partial_2 n_2, \ddot\partial_2\ell_2)} \\
&= (n_1{}^{\dot\partial_2 n_2}, (\ddot\partial_2\ell_2 \boxtimes n_1{}^{\dot\partial_2 n_2})^{-1}\ell_1{}^{(\dot\partial_2 n_2)(\ddot\partial_2\ell_2)}) \\
&= (n_1{}^{n_2}, \{(\ddot\partial_2\ell_2 \boxtimes n_2)(\ddot\partial_2\ell_2 \boxtimes n_1)^{n_2}(\ddot\partial_2\ell_2 \boxtimes n_2^{-1})^{n_1 n_2}\}^{-1}\ell_1{}^{(\dot\partial_2 n_2)(\ddot\partial_2\ell_2)}) \\
&= (n_1{}^{n_2}, ((\ell_2^{-1}\ell_2{}^{\dot\partial_2 n_2})(\ell_2^{-1}\ell_2{}^{\dot\partial_2 n_1})^{\dot\partial_2 n_2}(\ell_2^{-1}\ell_2{}^{\dot\partial_2 n_2^{-1}})^{\dot\partial_2(n_1 n_2)})^{-1}\ell_1{}^{(\dot\partial_2 n_2)(\ddot\partial_2\ell_2)} \\
&= (n_1{}^{n_2}, ((\ell_2^{-1})^{n_2^{-1}n_1 n_2}\ell_2)(\ell_2^{-1}\ell_1{}^{\dot\partial_2 n_2}\ell_2)) \\
&= (n_1{}^{n_2}, (\ell_2^{-1})^{n_2^{-1}n_1 n_2}\ell_1{}^{n_2}\ell_2) \\
&= (n_2^{-1}n_1 n_2, (\ell_2^{-1})^{n_2^{-1}n_1}\ell_1)(n_2,\ell_2) \\
&= (n_2,\ell_2)^{-1}(n_1,\ell_1)(n_2,\ell_2)
\end{aligned}$$

□

We may then construct a *cat*^{1}*-group of cat*^{1}*-groups* where the required homomorphisms are:

$$\begin{aligned}
\ddot t_2, \ddot h_2\ :\ (R_\emptyset \ltimes R_{\{1\}}) \ltimes (R_{\{2\}} \ltimes R_{[2]}) &\to R_\emptyset \ltimes R_{\{1\}} && (38) \\
\ddot t_2((p,m),(n,l)) &= (p,m) \\
\ddot h_2((p,m),(n,l)) &= (p,m)\,\bar\partial_2(n,l)\ =\ (p,m)(\dot\partial_2 n, \ddot\partial_2\ell)\ =\ (p(\dot\partial_2 n),\ m^{\dot\partial_2 n}(\ddot\partial_2\ell)) \\
\text{and}\qquad \ddot e_2\ :\ R_\emptyset \ltimes R_{\{1\}} &\to (R_\emptyset \ltimes R_{\{1\}}) \ltimes (R_{\{2\}} \ltimes R_{[2]}) && (39) \\
\ddot e_2(p,m) &= ((p,m),(1,1))\ .
\end{aligned}$$

We now check that $(\ddot e_2; \ddot t_2, \ddot h_2 : (R_\emptyset \ltimes R_{\{1\}}) \ltimes (R_{\{2\}} \ltimes R_{[2]}) \to R_\emptyset \ltimes R_{\{1\}})$ is a cat^1-group. Note that

$$\begin{aligned}
\ker \ddot t_2 &= \{\ ((1,1),(n,\ell))\ \} \\
\ker \ddot h_2 &= \{\ ((p,m),(n,\ell))\ \} \quad \text{where} \quad p = (\dot\partial_2 n)^{-1} \ \text{ and } \ m = ((\ddot\nu\ell)^{-1})^p,
\end{aligned}$$

so that

$$\ell^{pn} \;=\; \ell^{(\dot\partial_2 n)^{-1}(\dot\partial_2 n)} \;=\; \ell \qquad \text{in} \quad \ker \ddot h_2 .$$

The formula for the action is given by equation (36):

$$(n,\ell)^{(p,m)} \;:=\; (\; n^p,\; (m \boxtimes n^p)^{-1}\; \ell^{pm} \;)\,.$$

We now check the cat[1]-group axioms.

C1:

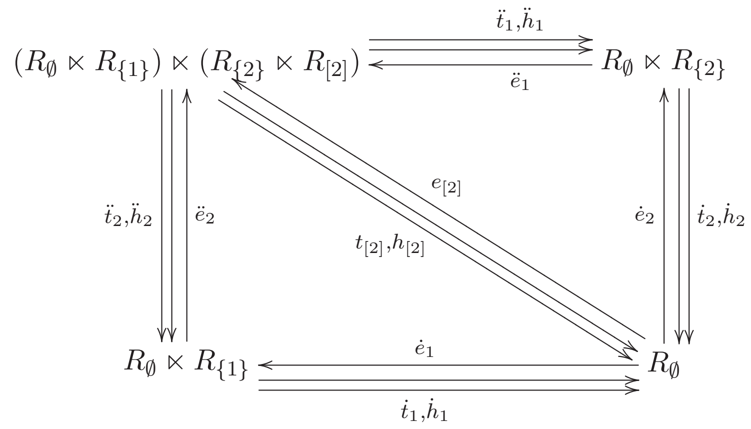
$$\begin{array}{rcccl} \ddot t_2 \ddot e_2(p,m) & = & \ddot t_2((p,m),(1,1)) & = & (p,m) \\ \ddot h_2 \ddot e_2(p,m) & = & \ddot h_2((p,m),(1,1)) & = & (p,m) \end{array}$$

C2:

$$\begin{array}{rcl} ((1,1),(n_0,\ell_0))\,((p,m),(n,\ell)) & = & ((p,m),(n_0,\ell_0)^{(p,m)}(n,\ell)) \\ & = & ((p,m),({n_0}^p,(m \boxtimes {n_0}^p)^{-1}{\ell_0}^{pm})(n,\ell)) \\ & = & ((p,m),({n_0}^{(\dot\partial_2 n)^{-1}}n,((m \boxtimes {n_0}^p)^{-1})^n {\ell_0}^{pmn}\ell)) \\ & = & ((p,m),(nn_0n^{-1}n,((\ddot\partial_2 \ell^{-1} \boxtimes n_0)^{-1})^{pn}\, {\ell_0}^{pmn}\ell)) \\ & = & ((p,m),(nn_0,(((\ell^{-1})^{-1}(\ell^{-1})^{n_0})^{-1})^{pn}){\ell_0}^{pmn}\ell)) \\ & = & ((p,m),(nn_0,(\ell^{n_0}\ell^{-1})^{pn}\, {\ell_0}^{pmn}\ell)) \\ & = & ((p,m),(nn_0,(\ell^{n_0}\ell^{-1})\, {\ell_0}^{p(\ddot\partial_2\ell^{-1})^p n}\ell)) \\ & = & ((p,m),(nn_0,(\ell^{n_0}\ell^{-1})\, {\ell_0}^{(\ddot\partial_2\ell^{-1})}\ell)) \\ & = & ((p,m),(nn_0,(\ell^{n_0}\ell^{-1})\, \ell\ell_0\ell^{-1}\ell)) \\ & = & ((p,m),(nn_0,\ell^{n_0}\,\ell_0)) \\ & = & ((p,m),(n,\ell)(n_0,\ell_0)) \\ & = & ((p,m),(n,\ell))\,((1,1),(n_0,\ell_0)) \end{array}$$

Theorem 5.16 *The homomorphisms in (38) and (39) give a cat[2]-group:*

$\mathcal{C}(\mathcal{R}) \quad =$



(40)

Proof: To be added. (Is material required from Section 7.2 ?) □

5.5 The other cat^2-structure

Now the underlying diagram (30) of the crossed square $\mathcal{R}$, together with the crossed pairing

$$\tilde{\boxtimes} \ : R_{\{2\}} \times R_{\{1\}} \to R_{[2]}, \ (n \mathbin{\tilde{\boxtimes}} m) = (m \boxtimes n)^{-1}$$

forms a second crossed square $\tilde{\mathcal{R}}$. Thus we can form a second cat^2-group $\mathcal{C}(\tilde{\mathcal{R}})$ with

$$\ddot{t}_1, \ddot{h}_1 : (R_\emptyset \ltimes R_{\{2\}}) \ltimes (R_{\{1\}} \ltimes R_{[2]}) \to (R_\emptyset \ltimes R_{\{2\}}) \ .$$

Let

$$\tilde{C} = (R_\emptyset \ltimes R_{\{2\}}) \ltimes (R_{\{1\}} \ltimes R_{[2]}), \qquad C = (R_\emptyset \ltimes R_{\{1\}}) \ltimes (R_{\{2\}} \ltimes R_{[2]}),$$

and

$$P = R_\emptyset \ltimes R_{\{2\}} \ , \quad \tilde{P} = R_\emptyset \ltimes R_{\{1\}} \ .$$

Proposition 5.17 *There is an isomorphism between these two semidirect products:*

$$\begin{aligned} \tau \ &: \ C \to \tilde{C}, \qquad ((p,m),(n,\ell)) \mapsto ((p,n),(m,(m \boxtimes n)\ell)) \ , \\ \tilde{\tau} \ := \ \tau^{-1} \ &: \ \tilde{C} \to C, \qquad ((p,n),(m,\ell)) \mapsto ((p,m),(n,(n \mathbin{\tilde{\boxtimes}} m)\ell)) \ . \end{aligned} \tag{41}$$

Proof:

$$\begin{aligned} & \tau((p_1,m_1),(n_1,\ell_1))((p_2,m_2),(n_2,\ell_2)) \\ = \ & \tau((p_1,m_1)(p_2,m_2),(n_1,\ell_1)^{(p_2,m_2)}(n_2,\ell_2)) \\ = \ & \tau((p_1p_2,{m_1}^{p_2}m_2),({n_1}^{p_2},(m_2 \boxtimes {n_1}^{p_2})^{-1}{\ell_1}^{p_2m_2})(n_2,\ell_2)) \\ = \ & \tau((p_1p_2,{m_1}^{p_2}m_2),({n_1}^{p_2}n_2,((m_2 \boxtimes {n_1}^{p_2})^{-1})^{n_2}{\ell_1}^{p_2m_2n_2}\ell_2)) \\ = \ & ((p_1p_2,{n_1}^{p_2}n_2),({m_1}^{p_2}m_2,({m_1}^{p_2}m_2 \boxtimes {n_1}^{p_2}n_2)((m_2 \boxtimes {n_1}^{p_2})^{-1})^{n_2}({\ell_1}^{p_2})^{m_2n_2}\ell_2)) \\ = \ & ((p_1p_2,{n_1}^{p_2}n_2),({m_1}^{p_2},({m_1}^{p_2} \boxtimes n_2)(m_1 \boxtimes n_1)^{p_2n_2}{\ell_1}^{p_2n_2})(m_2,(m_2 \boxtimes n_2)\ell_2)) \\ = \ & ((p_1,n_1)(p_2,n_2),(m_1,(m_1 \boxtimes n_1)\ell_1)^{(p_2,n_2)}(m_2,(m_2 \boxtimes n_2)\ell_2)) \\ = \ & ((p_1,n_1),(m_1,(m_1 \boxtimes n_1)\ell_1))((p_2,n_2),(m_2,(m_2 \boxtimes n_2)\ell_2)) \\ = \ & \tau((p_1,m_1),(n_1,\ell_1)) \ \tau((p_2,m_2),(n_2,\ell_2)) \end{aligned}$$

□

Note that the subgroup $(1 \ltimes R_{\{1\}}) \ltimes (R_{\{2\}} \ltimes 1)$ does *not* in general get mapped by τ to the subgroup $(1 \ltimes R_{\{2\}}) \ltimes (R_{\{1\}} \ltimes 1)$.

The isomorphism τ provides a neat formula for the inverse of a general element in $(P \ltimes N) \ltimes (M \ltimes L)$.

Lemma 5.18

$$((p,n),(m,\ell))^{-1} \ = \ \tau((p^{-1},(m^{-1})^{p^{-1}}), \ ((n^{-1})^{p^{-1}},(\ell^{-1})^{m^{-1}n^{-1}p^{-1}}))$$

Proof:

$$\begin{aligned} ((p,n),(m,\ell))^{-1} \ &= \ ((p^{-1},(n^{-1})^{p^{-1}}), \ (m^{-1},(\ell^{-1})^{m^{-1}})^{(p^{-1},(n^{-1})^{p^{-1}})})) \\ &= \ ((p^{-1},(n^{-1})^{p^{-1}}), \ ((m^{-1})^{p^{-1}},((m^{-1})^{p^{-1}} \boxtimes (n^{-1})^{p^{-1}})(\ell^{-1})^{m^{-1}n^{-1}p^{-1}})) \\ &= \ \tau((p^{-1},(m^{-1})^{p^{-1}}), \ ((n^{-1})^{p^{-1}},(\ell^{-1})^{m^{-1}n^{-1}p^{-1}}))) \end{aligned}$$

□

Lemma 5.19 *The commutator of elements in* $\ddot{e}_1(P \ltimes N)$ *and* $\ddot{e}_2(P \ltimes M)$ *is*

$$[((p,n),(1,1)),\ ((q,1),(m,1))] \ = \ (([p,q],n^{-1}n^q),((m^{-1})^{q^{-1}pq}m,((m^{-1})^{q^{-1}pq} \boxtimes n^q)^m))$$

Proof:

$$\begin{aligned} & [((p,n),(1,1)),\ ((q,1),(m,1))] \\ = \ & ((p^{-1},(n^{-1})^{p^{-1}}),(1,1))\ ((q^{-1},1),((m^{-1})^{q^{-1}},1))\ ((p,n),(1,1))\ ((q,1),(m,1)) \\ = \ & ((p^{-1}q^{-1},(n^{-1})^{p^{-1}q^{-1}}),((m^{-1})^{q^{-1}},1))\ ((pq,n^q),(m,1)) \\ = \ & ((p^{-1}q^{-1}pq,n^{-1}n^q),((m^{-1})^{q^{-1}pq},((m^{-1})^{q^{-1}pq} \boxtimes n^q))(m,1)) \\ = \ & (([p,q],n^{-1}n^q),((m^{-1})^{q^{-1}pq}m,((m^{-1})^{q^{-1}pq} \boxtimes n^q)^m)) \end{aligned}$$

□

6 Double Categories and Double Groupoids

Our interest here is in double groupoids and their connection with crossed squares and cat^2-groups.

6.1 Double Categories

A *double category* $\mathcal{D}$ consists of four sets and four category structures, and satisfies an interchange law:

- $D_{[2]}$, a set of *squares*,
- $D_{\{2\}}$, a set of *vertical* or *up-down arrows*,
- $D_{\{1\}}$, a set of *horizontal* or *left-right arrows*,
- $D_{\emptyset}$, a set of *objects*,
- maps $\ddot{t}_1, \ddot{h}_1 : D_{[2]} \to D_{\{2\}}$ and $\ddot{e}_1 : D_{\{2\}} \to D_{[2]}$ and a composition $\ddot{*}_1$ giving a category structure $\ddot{\mathcal{D}}_1$ on squares displayed horizontally,
- maps $\ddot{t}_2, \ddot{h}_2 : D_{[2]} \to D_{\{1\}}$ and $\ddot{e}_2 : D_{\{1\}} \to D_{[2]}$ and a composition $\ddot{*}_2$ giving a category structure $\ddot{\mathcal{D}}_2$ on squares displayed vertically,
- maps $\dot{t}_1, \dot{h}_1 : D_{\{1\}} \to D_{\emptyset}$, and $\dot{e}_1 : D_{\emptyset} \to D_{\{1\}}$ and a composition $\dot{*}_1$ giving a category structure $\dot{\mathcal{D}}_1$ on horizontal arrows,
- maps $\dot{t}_2, \dot{h}_2 : D_{\{2\}} \to D_{\emptyset}$, and $\dot{e}_2 : D_{\emptyset} \to D_{\{2\}}$ and a composition $\dot{*}_2$ giving a category structure $\dot{\mathcal{D}}_2$ on vertical arrows,
- the tail and head maps commute as follows:

$$\dot{t}_2\ddot{t}_1 = \dot{t}_1\ddot{t}_2, \quad \dot{t}_2\ddot{h}_1 = \dot{h}_1\ddot{t}_2, \quad \dot{h}_2\ddot{t}_1 = \dot{t}_1\ddot{h}_2, \quad \dot{h}_2\ddot{h}_1 = \dot{h}_1\ddot{h}_2,$$

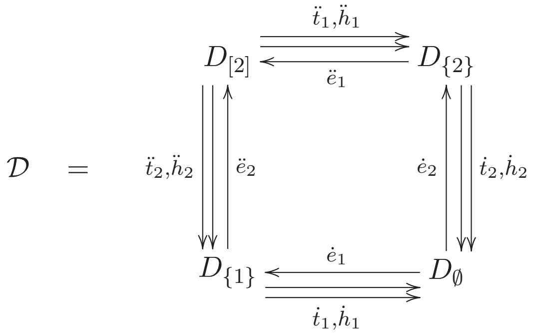
- for all squares A, B, C, D such that the compositions are defined,

$$\begin{array}{ccccc} \cdot & \longrightarrow & \cdot & \longrightarrow & \cdot \\ \downarrow & A & \downarrow & B & \downarrow \\ \cdot & \longrightarrow & \cdot & \longrightarrow & \cdot \\ \downarrow & C & \downarrow & D & \downarrow \\ \cdot & \longrightarrow & \cdot & \longrightarrow & \cdot \end{array} \qquad (A \mathbin{\ddot{*}_1} B) \mathbin{\ddot{*}_2} (C \mathbin{\ddot{*}_1} D) \;=\; (A \mathbin{\ddot{*}_2} C) \mathbin{\ddot{*}_1} (B \mathbin{\ddot{*}_2} D).$$

For a square A the four arrows and the four objects are displayed as follows.

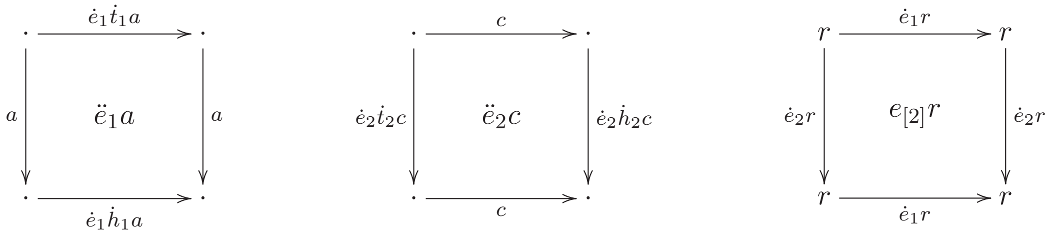
$$\begin{array}{ccc} \dot{t}_1\ddot{t}_2A & \xrightarrow{\ \ddot{t}_2A\ } & \dot{h}_1\ddot{t}_2A \\ {\scriptstyle \ddot{t}_1A}\Big\downarrow & A & \Big\downarrow{\scriptstyle \ddot{h}_1A} \\ \dot{t}_1\ddot{h}_2A & \xrightarrow[\ \ddot{h}_2A\ ]{} & \dot{h}_1\ddot{h}_2A \end{array}$$

The structure of a double category is shown in the following diagram:



(42)

Horizontal, vertical and double identity squares ($e_{[2]} = \ddot{e}_2\dot{e}_1 = \ddot{e}_1\dot{e}_2$) are represented by



Definition 6.1 *A morphism of double categories is a* double functor $\lambda : \mathcal{D} \to \mathcal{E}$*, given by a quadruple* $\lambda = (\lambda_{[2]}, \lambda_{\{2\}}, \lambda_{\{1\}}, \lambda_\emptyset)$ *of maps*

$$\lambda_{[2]} : D_{[2]} \to E_{[2]}, \quad \lambda_{\{2\}} : D_{\{2\}} \to E_{\{2\}}, \quad \lambda_{\{1\}} : D_{\{1\}} \to E_{\{1\}}, \quad \lambda_\emptyset : D_\emptyset \to E_\emptyset,$$

which combine to give functors from $\ddot{\mathcal{D}}_1, \ddot{\mathcal{D}}_2, \dot{\mathcal{D}}_1, \dot{\mathcal{D}}_2$ *to* $\ddot{\mathcal{E}}_1, \ddot{\mathcal{E}}_2, \dot{\mathcal{E}}_1, \dot{\mathcal{E}}_2$ *respectively, and which preserve all compositions.*

[Expand on this?]

6.2 Double Groupoids and Group – Double Groupoids

A *double groupoid* is a double category in which the four category structures are all groupoids. When the two sets of arrows coincide, $D_{\{1\}} = D_{\{2\}}$, we have an *arrow-symmetric double groupoid*. These are the subject of Chapter 6 of Brown and Sivera [18].

By analogy with the equivalence between cat^1-groups and group-groupoids, we attempt to describe an equivalence between cat^2-groups and group-double groupoids.

Definition 6.2 *A* group-double groupoid *is a double groupoid* $\mathcal{D}$ *where each of* $\mathcal{D}_{[2]}, \mathcal{D}_{\{2\}}, \mathcal{D}_{\{1\}}, \mathcal{D}_\emptyset$ *are groups and the twelve structure maps in (42) are group homomorphisms, so that the four category structures* $\ddot{\mathcal{D}}_1, \dot{\mathcal{D}}_1, \ddot{\mathcal{D}}_2, \dot{\mathcal{D}}_2$ *are all group-groupoids.*

[Is the diagonal a group-groupoid?]

6.3 Horizontal, Vertical and Double Sections

We saw in Subsections 1.13 and 2.3 that a section of a group-groupoid is a group monomorphism $\xi : G_0 \to G_1$ such that $t\xi = 1_{G_0}$. In order to generalise this to a horizontal section of a group-double groupoid $\mathcal{D}$, we require compatible monomorphisms from up-down arrows to squares and from points to left-right arrows. In order to see what 'compatible' means in this context we note that, in the following diagrams, the left-right boundaries of $\ddot{\xi}_1 g$ should be the images of the points of g under $\dot{\xi}_1$, while $\ddot{\xi}_1 \dot{e}_2 r$ should be the vertical identity square for $\dot{\xi}_1 r$. We show up-down arrows as dashed in the diagrams in this Subsection.

$$\begin{array}{ccc} \dot{t}_2 g & \xrightarrow{\dot{\xi}_1 \dot{t}_2 g = \ddot{t}_2 \ddot{\xi}_1 g} & \dot{t}_2 \ddot{h}_1 \ddot{\xi}_1 g \\ {\scriptstyle g} \downarrow & \ddot{\xi}_1 g & \downarrow {\scriptstyle \ddot{h}_1 \ddot{\xi}_1 g} \\ \dot{h}_2 g & \xrightarrow[\dot{\xi}_1 \dot{h}_2 g = \ddot{h}_2 \ddot{\xi}_1 g]{} & \dot{h}_2 \ddot{h}_1 \ddot{\xi}_1 g \end{array} \qquad \begin{array}{ccc} r & \xrightarrow{\dot{\xi}_1 r} & \dot{h}_1 \dot{\xi}_1 r \\ {\scriptstyle \dot{e}_2 r} \downarrow & \ddot{\xi}_1 \dot{e}_2 r = \ddot{e}_2 \dot{\xi}_1 r & \downarrow {\scriptstyle \ddot{h}_1 \ddot{\xi}_1 \dot{e}_2 r} \\ r & \xrightarrow[\dot{\xi}_1 r]{} & \dot{h}_1 \dot{\xi}_1 r \end{array}$$

Definition 6.3
(a) *A* horizontal section *of a group-double groupoid $\mathcal{D}$ is a pair $\xi_1 = (\ddot{\xi}_1, \dot{\xi}_1)$ of monomorphisms $\ddot{\xi}_1 : D_1 \to D_{[2]}$ and $\dot{\xi}_1 : D_\emptyset \to D_{\{2\}}$ such that*

$$\ddot{t}_1 \ddot{\xi}_1 = 1_{D_{\{1\}}}, \quad \dot{t}_1 \dot{\xi}_1 = 1_{D_\emptyset}, \quad \ddot{t}_2 \ddot{\xi}_1 g = \dot{\xi}_1 \dot{t}_2 g, \quad \ddot{h}_2 \ddot{\xi}_1 g = \dot{\xi}_1 \dot{h}_2 g, \quad \ddot{\xi}_1 \dot{e}_2 r = \ddot{e}_2 \dot{\xi}_1 r.$$

(b) *A* vertical section *of a group-double groupoid $\mathcal{D}$ is a pair $\xi_2 = (\ddot{\xi}_2, \dot{\xi}_2)$ of monomorphisms $\ddot{\xi}_2 : D_2 \to D_{[2]}$ and $\dot{\xi}_2 : D_\emptyset \to D_{\{1\}}$ such that*

$$\ddot{t}_2 \ddot{\xi}_2 = 1_{D_{\{2\}}}, \quad \dot{t}_2 \dot{\xi}_2 = 1_{D_\emptyset}, \quad \ddot{t}_1 \ddot{\xi}_2 g = \dot{\xi}_2 \dot{t}_1 g, \quad \ddot{h}_1 \ddot{\xi}_2 g = \dot{\xi}_2 \dot{h}_1 g, \quad \ddot{\xi}_2 \dot{e}_1 r = \ddot{e}_1 \dot{\xi}_2 r.$$

[Maybe we should swap 1 and 2 in ξ_1, ξ_2 ?]

Given a section $\xi_1 = (\ddot{\xi}_1, \dot{\xi}_1)$ of $\ddot{\mathcal{D}}_1$ we may apply the construction in Subsection 2.3 to obtain a groupoid automorphism λ_1 of $\ddot{\mathcal{D}}_1$, which extends to a double groupoid automorphisms $\lambda_1 : \mathcal{D} \to \mathcal{D}$.

$$\begin{array}{ccc} \dot{h}_1 \dot{\xi}_1 \dot{t}_1 \ddot{t}_2 A & \xrightarrow{\dot{\lambda}_1 \ddot{t}_2 A} & \dot{h}_1 \dot{\xi}_1 \dot{h}_1 \ddot{t}_2 A \\ {\scriptstyle \ddot{h}_1 \ddot{\xi}_1 \ddot{t}_1 A} \downarrow & \ddot{\lambda}_1 A & \downarrow {\scriptstyle \ddot{h}_1 \ddot{\xi}_1 \ddot{h}_1 A} \\ \dot{h}_1 \dot{\xi}_1 \dot{t}_1 \ddot{h}_2 A & \xrightarrow[\dot{\lambda}_1 \ddot{h}_2 A]{} & \dot{h}_1 \dot{\xi}_1 \dot{h}_1 \ddot{h}_2 A \end{array} \quad = \quad \begin{array}{ccccccc} \dot{h}_1 \dot{\xi}_1 \dot{t}_1 \ddot{t}_2 A & \xleftarrow{\dot{\xi}_1 \dot{t}_1 \ddot{t}_2 A} & \dot{t}_1 \ddot{t}_2 A & \xrightarrow{\ddot{t}_2 A} & \dot{h}_1 \ddot{t}_2 A & \xrightarrow{\dot{\xi}_1 \dot{h}_1 \ddot{t}_2 A} & \dot{h}_1 \dot{\xi}_1 \dot{h}_1 \ddot{t}_2 A \\ {\scriptstyle \ddot{h}_1 \ddot{\xi}_1 \ddot{t}_1 A} \downarrow & \ddot{\xi}_1 \ddot{t}_1 A & \downarrow {\scriptstyle \ddot{t}_1 A} & A & {\scriptstyle \ddot{h}_1 A} \downarrow & \ddot{\xi}_1 \ddot{h}_1 A & \downarrow {\scriptstyle \ddot{h}_1 \ddot{\xi}_1 \ddot{h}_1 A} \\ \dot{h}_1 \dot{\xi}_1 \dot{t}_1 \ddot{h}_2 A & \xleftarrow[\dot{\xi}_1 \dot{t}_1 \ddot{h}_2 A]{} & \dot{t}_1 \ddot{h}_2 A & \xrightarrow[\ddot{h}_2 A]{} & \dot{h}_1 \ddot{h}_2 A & \xrightarrow[\dot{\xi}_1 \dot{h}_1 \ddot{h}_2 A]{} & \dot{h}_1 \dot{\xi}_1 \dot{h}_1 \ddot{h}_2 A \end{array}$$

The defining equations for λ_1 are:

$$\begin{aligned} (\lambda_1)_{[2]}\, A &= \widetilde{(\ddot{\xi}_1 \ddot{t}_1 A)} \;\ddot{*}_1\; A \;\ddot{*}_1\; (\ddot{\xi}_1 \ddot{h}_1 A)\,, \qquad (43) \\ (\lambda_1)_{\{2\}}\, a &= \ddot{h}_1 \ddot{\xi}_1\, a\,, \\ (\lambda_1)_{\{1\}}\, c &= \widetilde{(\dot{\xi}_1 \dot{t}_1 c)} \;\dot{*}_1\; c \;\dot{*}_1\; (\dot{\xi}_1 \dot{h}_1 c), \\ (\lambda_1)_\emptyset\, p &= \dot{h}_1 \dot{\xi}_1\, p\,. \end{aligned}$$

Similarly, given a section $\xi_2 = (\ddot{\xi}_2, \dot{\xi}_2)$ of $\ddot{\mathcal{D}}_2$, applying the same construction we obtain

$$
\begin{array}{ccc}
\dot h_2\dot\xi_2\dot t_1\ddot t_2 A & \xrightarrow{\ddot h_2\ddot\xi_2\ddot t_2 A} & \dot h_2\dot\xi_2\dot h_1\ddot t_2 A \\
\Big\downarrow{\scriptstyle \dot\lambda_2\dot t_1 A} & \ddot\lambda_2 A & \Big\downarrow{\scriptstyle \dot\lambda_2\dot h_1 A} \\
\dot h_2\dot\xi_2\dot t_1\ddot h_2 A & \xrightarrow[\ddot h_2\ddot\xi_2\ddot h_2 A]{} & \dot h_2\dot\xi_2\dot h_1\ddot h_2 A
\end{array}
\quad = \quad
\begin{array}{ccc}
\dot h_2\dot\xi_2\dot t_1\ddot t_2 A & \xrightarrow{\ddot h_2\ddot\xi_2\ddot t_2 A} & \dot h_2\dot\xi_2\dot h_1\ddot t_2 A \\
\Big\uparrow{\scriptstyle \dot\xi_2\dot t_1\ddot t_2 A} & \ddot\xi_2\ddot t_2 A & \Big\uparrow{\scriptstyle \dot\xi_2\dot h_1\ddot t_2 A} \\
\dot t_1\ddot t_2 A & \xrightarrow[\ddot t_2 A]{} & \dot h_1\ddot t_2 A \\
\Big\downarrow{\scriptstyle \dot t_1 A} & A & \Big\downarrow{\scriptstyle \dot h_1 A} \\
\dot t_1\ddot h_2 A & \xrightarrow{\ddot h_2 A} & \dot h_1\ddot h_2 A \\
\Big\downarrow{\scriptstyle \dot\xi_2\dot t_1\ddot h_2 A} & \ddot\xi_2\ddot h_2 A & \Big\downarrow{\scriptstyle \dot\xi_2\dot h_1\ddot h_2 A} \\
\dot h_2\dot\xi_2\dot t_1\ddot h_2 A & \xrightarrow[\ddot h_2\ddot\xi_2\ddot h_2 A]{} & \dot h_2\dot\xi_2\dot h_1\ddot h_2 A
\end{array}
$$

This determines a double groupoid automorphism λ_2 of $\mathcal{D}$ where

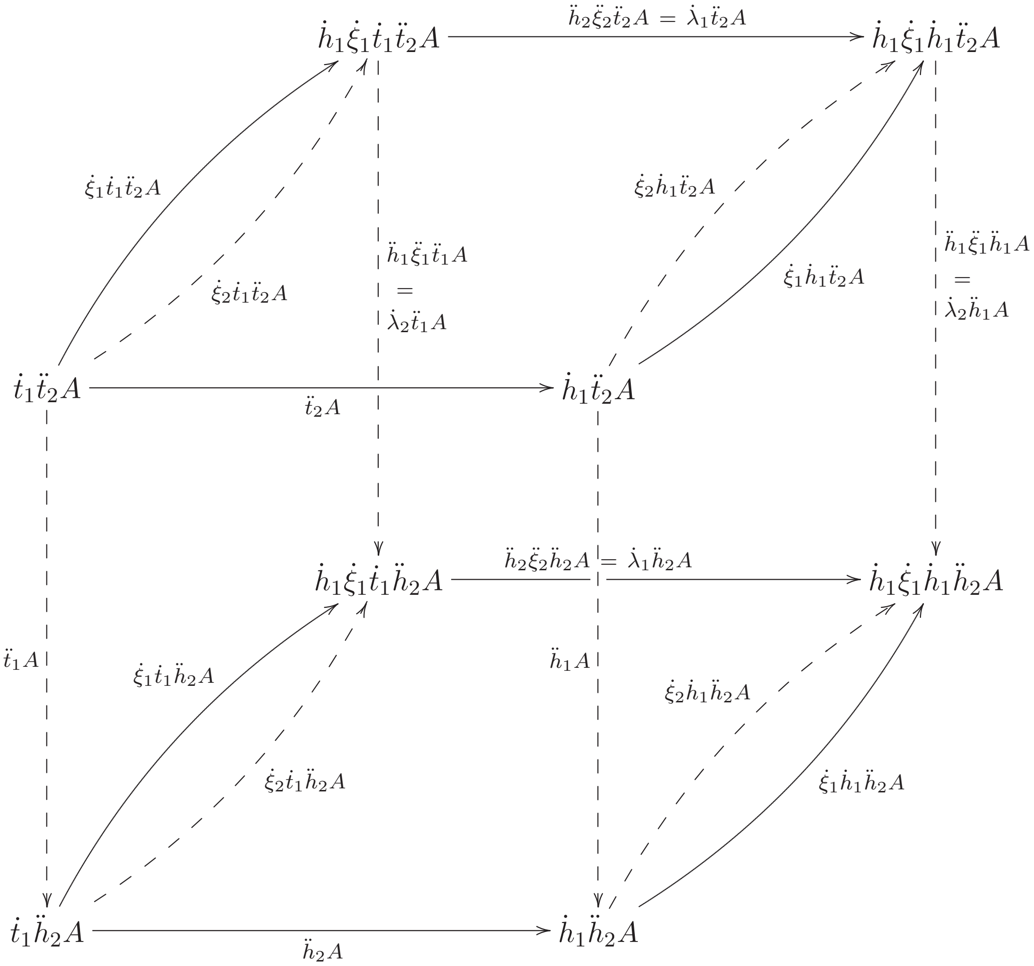
$$
\begin{aligned}
(\lambda_2)_{[2]}\, A &= (\widetilde{\ddot\xi_2\ddot t_2 A}) \mathbin{\ddot{*}_2} A \mathbin{\ddot{*}_2} (\ddot\xi_2\ddot h_2 A)\,, \\
(\lambda_2)_{\{2\}}\, a &= (\widetilde{\dot\xi_2\dot t_2 a}) \mathbin{\dot{*}_2} a \mathbin{\dot{*}_2} (\dot\xi_2\dot h_2 a)\,, \\
(\lambda_2)_{\{1\}}\, c &= \ddot h_2\ddot\xi_2\, c\,, \\
(\lambda_2)_{\emptyset}\, p &= \dot h_2\dot\xi_2\, p\,.
\end{aligned}
\tag{44}
$$

Given a pair of sections $\xi = (\xi_1, \xi_2)$, one horizontal and one vertical, if we apply both constructions we obtain an automorphism $\lambda = \lambda_\xi$ of $\mathcal{D}$ provided $\lambda_1 A = \lambda_2 A$. We call ξ a *double section* of $\mathcal{D}$. The requirement $\lambda_1 A = \lambda_2 A$ implies four equations at the four levels of $\mathcal{D}$,

$$
\begin{aligned}
\dot h_1\dot\xi_1\, p &= \dot h_2\dot\xi_2\, p\,, \\
\ddot h_1\ddot\xi_1\, a &= (\widetilde{\dot\xi_2\dot t_2 a}) \mathbin{\dot{*}_2} a \mathbin{\dot{*}_2} (\dot\xi_2\dot h_2 a) \;=\; \dot\lambda_2\, a\,, \\
\dot\lambda_1\, c \;=\; (\widetilde{\dot\xi_1\dot t_1 c}) \mathbin{\dot{*}_1} c \mathbin{\dot{*}_1} (\dot\xi_1\dot h_1 c) &= \ddot h_2\ddot\xi_2\, c\,, \\
\ddot\lambda_1\, A \;=\; (\widetilde{\ddot\xi_1\ddot t_1 A}) \mathbin{\ddot{*}_1} A \mathbin{\ddot{*}_1} (\ddot\xi_1\ddot h_1 A) &= (\widetilde{\ddot\xi_2\ddot t_2 A}) \mathbin{\ddot{*}_2} A \mathbin{\ddot{*}_2} (\ddot\xi_2\ddot h_2 A) \;=\; \ddot\lambda_2\, A\,.
\end{aligned}
$$

The previous two diagrams may be combined into the cube-like figure of equation (45), where the six square faces are denoted F=front, L=left, R=right, U=up, D=down and B=back, and

$$
F = A, \quad L = \ddot\xi_1\ddot t_1 A, \quad R = \ddot\xi_1\ddot h_1 A, \quad U = \ddot\xi_2\ddot t_2 A, \quad D = \ddot\xi_2\ddot h_2 A, \quad B = \lambda A.
$$



$\dot{h}_1\dot{\xi}_1\dot{t}_1\ddot{t}_2A$
$\ddot{h}_2\ddot{\xi}_2\ddot{t}_2A = \dot{\lambda}_1\ddot{t}_2A$
$\dot{h}_1\dot{\xi}_1\dot{h}_1\ddot{t}_2A$
$\dot{\xi}_1\dot{t}_1\ddot{t}_2A$
$\dot{\xi}_2\dot{t}_1\ddot{t}_2A$
$\ddot{h}_1\ddot{\xi}_1\ddot{t}_1A = \dot{\lambda}_2\ddot{t}_1A$
$\dot{\xi}_2\dot{h}_1\ddot{t}_2A$
$\dot{\xi}_1\dot{h}_1\ddot{t}_2A$
$\ddot{h}_1\ddot{\xi}_1\ddot{h}_1A = \dot{\lambda}_2\ddot{h}_1A$
$\dot{t}_1\ddot{t}_2A$
$\ddot{t}_2A$
$\dot{h}_1\ddot{t}_2A$
$\dot{h}_1\dot{\xi}_1\dot{t}_1\ddot{h}_2A$
$\ddot{h}_2\ddot{\xi}_2\ddot{h}_2A = \dot{\lambda}_1\ddot{h}_2A$
$\dot{h}_1\dot{\xi}_1\dot{h}_1\ddot{h}_2A$
$\ddot{t}_1A$
$\dot{\xi}_1\dot{t}_1\ddot{h}_2A$
$\ddot{h}_1A$
$\dot{\xi}_2\dot{h}_1\ddot{h}_2A$
$\dot{\xi}_2\dot{t}_1\ddot{h}_2A$
$\dot{\xi}_1\dot{h}_1\ddot{h}_2A$
$\dot{t}_1\ddot{h}_2A$
$\ddot{h}_2A$
$\dot{h}_1\ddot{h}_2A$

(45)

7 2-crossed modules

The reader is refered to Brown–Gilbert [7], Conduché [25],[26], and Mutlu–Porter [46] for background information on 2-crossed modules.

Definition 7.1 *A 2-crossed module is comprised of the following:*

- *a 2-complex of groups*

$$\mathcal{Z} \quad = \quad (\, T \xrightarrow{\delta_2} U \xrightarrow{\delta_1} V \,)$$

  (so that $\delta_1 \circ \delta_2 = 0 : T \to V$);

- *an action of V on T and on U, and on itself by conjugation, such that δ_1, δ_2 are morphisms which preserve the actions;*

- *a function, called the* Peiffer lifting*,*

$$\{\ ,\ \} \ : \ U \times U \to T,$$

  making $(\delta_2 : T \to U)$ a crossed module with action

$$t^u \ := \ t\,\{\delta_2 t, u\}\ . \tag{46}$$

The following axioms are also required:

(2X1) $\delta_2\{u_1, u_2\} = \langle u_1, u_2\rangle = u_1^{-1} u_2^{-1} u_1 u_2{}^{\delta_1 u_1}$ *(a Peiffer commutator),*

(2X2) $\{u, \delta_2 t\} = (t^{-1})^u\, t^{\delta_1 u}$,

(2X3) $\{u_1 u_2, u_3\} = \{u_1, u_3\}^{u_2}\{u_2, u_3{}^{\delta_1 u_1}\}$,

(2X4) $\{u_1, u_2 u_3\} = \{u_1, u_3\}\{u_1, u_2\}^{u_3{}^{\delta_1 u_1}}$,

(2X5) $\{u_1, u_2\}^v = \{u_1{}^v, u_2{}^v\}$.

An additional axiom, $\{\delta_2 t, u\} = t^{-1} t^n$ id often specified, but we have used this identity in (46) to define the action. Note that δ_2 maps (2X3), (2X4) and (2X5) to identities (b), (c) and (d) in Lemma 1.10 for Peiffer commutators. Compare also (2X3), (2X4) with identities (a), (b) for crossed pairings in Definition 4.4.

We will check that the crossed module action $t^u = t\,\{\delta_2 t, u\}$ given by formula (46) is well defined:

$$\begin{aligned}
(t_1 t_2)^u &= t_1 t_2 \{(\delta_2 t_1)(\delta_2 t_2), u\} \\
&= t_1 t_2 \{\delta_2 t_1, u\}^{\delta_2 t_2}\{\delta_2 t_2, u^{\delta_1 \delta_2 t_1}\} \\
&= t_1 t_2 t_2^{-1}\{\delta_2 t_1, u\} t_2 \{\delta_2 t_2, u\} \\
&= t_1{}^u\, t_2{}^u, \\
t^{(u_1 u_2)} &= t\{\delta_2 t, u_1 u_2\} \\
&= t\{\delta_2 t, u_2\}\{\delta_2 t, u_1\}^{u_2{}^{\delta_1 \delta_2 t}} \\
&= t^{u_2}\{\delta_2 t, u_1\}^{u_2} \\
&= (t^{u_1})^{u_2}.
\end{aligned}$$

Lemma 7.2 (a) $\{\delta_2 t_1, \delta_2 t_2\} = [t_1, t_2]$.

Proof:

(a) $\{\delta_2 t_1, \delta_2 t_2\} = t_1^{-1}\, {t_1}^{\delta_2 t_2} = [t_1, t_2]$, by definition of the crossed module action.

□

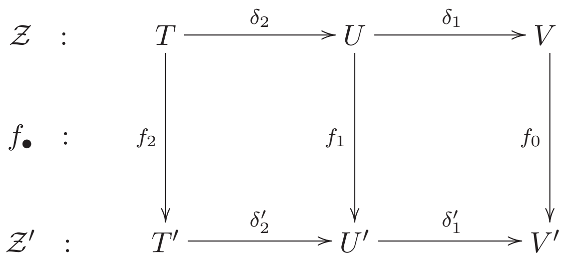
7.1 Morphisms and Homotopies of 2-crossed modules

Definition 7.3 *A morphism of* 2*-crossed modules is a triple of group homomorphisms*

$$f_\bullet \;\equiv\; (f_2, f_1, f_0) \;:\; \mathcal{Z} \to \mathcal{Z}'$$

such that

$$f_1\delta_2 = \delta_2' f_2, \quad f_0\delta_1 = \delta_1' f_1, \quad f_2(t^v) = (f_2 t)^{f_0 v}, \quad f_1(u^v) = (f_1 u)^{f_0 v}, \quad f_2\{u_1, u_2\} = \{f_1 u_1, f_1 u_2\}.$$



An automorphism of $\mathcal{Z}$ is an endomorphism with inverse ${f_\bullet}^{-1} = (f_2^{-1}, f_1^{-1}, f_0^{-1})$.

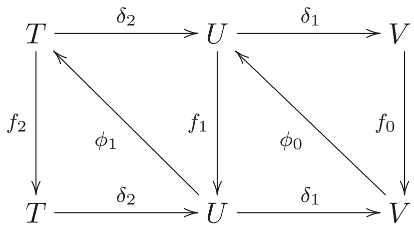
Note that

$$f_2(t^u) \;=\; (f_2 t)\{f_1\delta_2 t, f_1 u\} \;=\; (f_2 t)\{\delta_2' f_2 t, f_1 u\} \;=\; (f_2 t)^{f_1 u}$$

so $(f_2, f_1) : (\delta_2 : T \to U) \to (\delta_2' : T' \to U')$ is a morphism of crossed modules.

Definition 7.4 *A homotopy of the* 2*-crossed module* $\mathcal{Z}$ *is a pair of homomorphisms* $\phi_\bullet = (\phi_1, \phi_0)$ *such that*

$$\delta_2\phi_1 \;=\; \phi_0\delta_1 \qquad \textbf{[Is that all ???]}$$



7.2 2-crossed modules of groupoids

[The rest of this section really belongs in a crossed squares chapter.]

7.3 The 2-crossed module associated to a crossed square

We follow Brown–Gilbert [7] in defining the appropriate Peiffer lifting to be $\{(m_1, n_1), (m_2, n_2)\} = {m_2}^{m_1} \boxtimes n_1$.

Proposition 7.5 *Given a crossed square $\mathcal{R}$ there is an associated 2-crossed module $\mathcal{Z}$, as shown in the following diagram:*

$$\mathcal{R} \;:\; \begin{array}{ccc} R_{[2]} & \xrightarrow{\ddot{\partial}_1} & R_{\{2\}} \\ {\scriptstyle \ddot{\partial}_2}\big\downarrow & & \big\downarrow{\scriptstyle \dot{\partial}_2} \\ R_{\{1\}} & \xrightarrow[\dot{\partial}_1]{} & R_{\emptyset} \end{array} \qquad\qquad \mathcal{Z} \;:\; \left(R_{[2]} \xrightarrow{\delta_2} R_{\{1\}} \ltimes R_{\{2\}} \xrightarrow{\delta_1} R_{\emptyset} \right)$$

where

$$\delta_2 \ell = (\ddot{\partial}_2 \ell, \ddot{\partial}_1 \ell^{-1}) \qquad \textit{and} \qquad \delta_1(m, n) = (\dot{\partial}_1 m)(\dot{\partial}_2 n) \, ,$$

$R_{\emptyset}$ acts diagonally on $R_{\{1\}} \ltimes R_{\{2\}}$, and the Peiffer lifting is given by

$$\{(m_1, n_1), (m_2, n_2)\} \;=\; {m_2}^{m_1} \boxtimes n_1 \, .$$

Proof: We first check that δ_1 and δ_2 are homomorphisms preserving the $R_{\emptyset}$-actions, and that $\delta_1 \delta_2 = 0$:

$$\begin{array}{rcl}
\delta_1((m_1, n_1), (m_2, n_2)) &=& \delta_1(m_1 m_2, {n_1}^{m_2} n_2) \;=\; \dot{\partial}_1(m_1 m_2)\, \dot{\partial}_2({n_1}^{\dot{\partial}_1 m_2} n_2) \\
&=& (\dot{\partial}_1 m_1)(\dot{\partial}_1 m_2)(\dot{\partial}_2 n_1)^{\dot{\partial}_1 m_2}(\dot{\partial}_2 n_2) \;=\; \delta_1(m_1, n_1)\, \delta_1(m_2, n_2). \\
\delta_1((m, n)^p) &=& \delta_1(m^p, n^p) \;=\; \dot{\partial}_1(m^p)\, \dot{\partial}_2(n^p) \;=\; (\dot{\partial}_1 m)^p\, (\dot{\partial}_2 m)^p \;=\; (\delta_1(m, n))^p \, . \\
(\delta_2 \ell_1)(\delta_2 \ell_2) &=& (\ddot{\partial}_2 \ell_1, \ddot{\partial}_1 \ell_1^{-1})(\ddot{\partial}_2 \ell_2, \ddot{\partial}_1 \ell_2^{-1}) \\
&=& ((\ddot{\partial}_2 \ell_1)(\ddot{\partial}_2 \ell_2), (\ddot{\partial}_1 \ell_1^{-1})^{\dot{\partial}_1 \ddot{\partial}_2 \ell_2}(\ddot{\partial}_1 \ell_2^{-1})) \\
&=& (\ddot{\partial}_2(\ell_1 \ell_2), (\ddot{\partial}_1 \ell_1^{-1})^{\dot{\partial}_2 \ddot{\partial}_1 \ell_2}(\ddot{\partial}_1 \ell_2^{-1})) \\
&=& (\ddot{\partial}_2(\ell_1 \ell_2), (\ddot{\partial}_1 \ell_2^{-1})(\ddot{\partial}_1 \ell_1^{-1})) \qquad\qquad \text{by } \mathbf{X2:} \text{ for } \dot{\mathcal{R}}_2 \\
&=& \delta_2(\ell_1 \ell_2). \\
\delta_2(\ell^p) &=& (\ddot{\partial}_2(\ell^p), \ddot{\partial}_1((\ell^p)^{-1})) \;=\; ((\ddot{\partial}_2 \ell)^p, (\ddot{\partial}_1 \ell^{-1})^p) \;=\; (\ddot{\partial}_2 \ell, \ddot{\partial}_1 \ell^{-1})^p \;=\; (\delta_2 \ell)^p \, . \\
\delta_1 \delta_2 \ell &=& \delta_1(\ddot{\partial}_2 \ell, \ddot{\partial}_1 \ell^{-1}) \;=\; (\dot{\partial}_1 \ddot{\partial}_2 \ell)(\dot{\partial}_2 \ddot{\partial}_1 \ell^{-1}) \;=\; 1 \, .
\end{array}$$

Secondly we identify the crossed module action in $\mathcal{Z}$ in this case to be $\ell^{(m,n)} = \ell^m$.

$$\begin{aligned}
\ell^{(m,n)} &= \ell\,\{(\ddot{\partial}_2\ell, \ddot{\partial}_1\ell^{-1}), (m,n)\} \\
&= \ell\,(m^{\ddot{\partial}_2\ell} \boxtimes \ddot{\partial}_1\ell^{-1}) \\
&= \ell\,\ell^{(\ddot{\partial}_2\ell^{-1})m(\ddot{\partial}_2\ell)}\,\ell^{-1} && \text{by Definition 5.1 (e)} \\
&= \ell\,(\ell^m)^{(\ddot{\partial}_2\ell)}\,\ell^{-1} \;=\; \ell^m && \text{by } \mathbf{X2:} \text{ (twice).}
\end{aligned}$$

It is clear that this *is* an action, so we verify the two crossed module axioms:

$$\begin{aligned}
\mathbf{X1}: \quad (\delta_2\ell)^{(m,n)} &= (m^{-1}, (n^{-1})^{m^{-1}})(\ddot{\partial}_2\ell, \ddot{\partial}_1\ell^{-1})(m,n) \\
&= (m^{-1}(\ddot{\partial}_2\ell)m,\; (n^{-1})^{\dot{\partial}_1\ddot{\partial}_2\ell^m}(\ddot{\partial}_1\ell^{-1})^m n) \\
&= ((\ddot{\partial}_2\ell)^m,\; (n^{-1})^{\dot{\partial}_2(\ddot{\partial}_1\ell)^m}(\ddot{\partial}_1\ell^m)^{-1}n) \\
&= ((\ddot{\partial}_2\ell)^m,\; (\ddot{\partial}_1\ell^m)^{-1}n^{-1}n) \\
&= \delta_2(\ell^m) \;=\; \delta_2(\ell^{(m,n)}), \\
\mathbf{X2}: \quad {\ell_0}^{\delta_2\ell} &= {\ell_0}^{(\ddot{\partial}_2\ell, \ddot{\partial}_1\ell^{-1})} \;=\; {\ell_0}^{\ddot{\partial}_2\ell} \;=\; {\ell_0}^{\ell}\,.
\end{aligned}$$

Thirdly, we verify the five axioms.

2X1:

$$\begin{aligned}
&(m_1,n_1)^{-1}\,(m_2,n_2)^{-1}\,(m_1,n_1)\,(m_2,n_2)^{\delta_1(m_1,n_1)} \\
=\;& ({m_1}^{-1}, ({n_1}^{-1})^{{m_1}^{-1}})\,({m_2}^{-1}, ({n_2}^{-1})^{{m_2}^{-1}})\,(m_1,n_1)\,(m_2,n_2)^{(\dot{\partial}_1 m_1)(\dot{\partial}_2 n_1)} \\
=\;& (m_1^{-1}m_2^{-1}m_1{m_2}^{m_1n_1}, (n_1^{-1})^{m_1^{-1}m_2^{-1}m_1{m_2}^{m_1n_1}}(n_2^{-1})^{m_2^{-1}m_1{m_2}^{m_1n_1}}{n_1}^{{m_2}^{m_1n_1}}{n_2}^{m_1n_1})
\end{aligned}$$

The left hand element is

$$({m_2}^{m_1})^{-1}\,({m_2}^{m_1})^{n_1} \;=\; \ddot{\partial}_2({m_2}^{m_1} \boxtimes n_1)$$

It follows that the right hand element is

$$\begin{aligned}
&(n_1^{-1}(n_2^{-1})^{m_1})^{\ddot{\partial}_2({m_2}^{m_1}\boxtimes n_1)}\,{n_1}^{{m_2}^{m_1n_1}}\,{n_2}^{m_1n_1} \\
=\;& (n_1^{-1}({n_2}^{m_1})^{-1})^{(n_1^{-1})^{{m_2}^{m_1}}n_1}\,({n_1}^{{m_2}^{m_1}})^{n_1}\,({n_2}^{m_1})^{n_1} && \text{by Lemmas 2.7(c) and 2.2(d),} \\
=\;& (n_1^{-1})({n_1}^{{m_2}^{m_1}})(n_1^{-1})({n_2}^{m_1})^{-1}({n_1}^{{m_2}^{m_1}})^{-1}(n_1)\,(n_1^{-1})({n_1}^{{m_2}^{m_1}})(n_1)\,(n_1^{-1})({n_2}^{m_1})(n_1) \\
=\;& \ddot{\partial}_1({m_2}^{m_1} \boxtimes n_1)^{-1}
\end{aligned}$$

So the pair of elements is $\delta_2({m_2}^{m_1} \boxtimes n_1) \;=\; \delta_2\{(m_1,n_1),(m_2,n_2)\}$.

2X2:

$$\{(m,n), \delta_2\ell\} \;=\; \{(m,n), (\ddot{\partial}_2\ell, \ddot{\partial}_1\ell^{-1})\} \;=\; \ddot{\partial}_2(\ell^m) \boxtimes n \;=\; (\ell^m)^{-1}(\ell^m)^n \;=\; (\ell^{-1})^{(m,n)}\ell^{\delta_1(m,n)}\,.$$

2X3:

$$\begin{aligned}
&\{(m_1,n_1),(m_3,n_3)\}^{(m_2,n_2)}\,\{(m_2,n_2),(m_3,n_3)^{m_1n_1}\} \\
=\;& ({m_3}^{m_1} \boxtimes n_1)^{m_2}\,({m_3}^{m_1n_1m_2} \boxtimes n_2) \\
=\;& (m_0 \boxtimes n_0)\,({m_0}^{n_0} \boxtimes n_2) && \text{where } m_0 = {m_3}^{m_1m_2},\; n_0 = {n_1}^{m_2} \\
=\;& (m_0 \boxtimes n_2)\,(m_0 \boxtimes n_0)^{n_2} && \text{by Proposition 4.6(d)} \\
=\;& m_0 \boxtimes n_0n_2 \\
=\;& {m_3}^{m_1m_2} \boxtimes {n_1}^{m_2}n_2 \\
=\;& \{(m_1m_2, {n_1}^{m_2}n_2), (m_3,n_3)\} \\
=\;& \{(m_1,n_1)(m_2,n_2), (m_3,n_3)\}
\end{aligned}$$

2X4:

$$\begin{aligned}
& \{(m_1,n_1),(m_2m_3,{n_2}^{m_3}n_3)\} \\
= \;& (m_2m_3)^{m_1} \boxtimes n_1 \\
= \;& ({m_2}^{m_1} \boxtimes n_1)^{{m_3}^{m_1}}\,({m_3}^{m_1} \boxtimes n_1) \\
= \;& (m_4 \boxtimes n_1)^{n_1m_5}\,(m_5 \boxtimes n_1) && \text{where } m_4 = {m_2}^{m_1n_1^{-1}},\; m_5 = {m_3}^{m_1} \\
= \;& (m_5 \boxtimes n_1)(m_4 \boxtimes n_1)^{m_5n_1} && \text{by Proposition 4.6(c)} \\
= \;& ({m_3}^{m_1} \boxtimes n_1)({m_2}^{m_1} \boxtimes n_1)^{n_1^{-1}{m_3}^{m_1}n_1} \\
= \;& \{(m_1,n_1),(m_3,n_3)\}\,\{(m_1,n_1),(m_2,n_2)\}^{(m_3,n_3)^{\delta_1(m_1,n_1)}}
\end{aligned}$$

2X5:

$$\{({m_1}^p,{n_1}^p),({m_2}^p,{n_2}^p)\} \;=\; ({m_2}^p)^{{m_1}^p} \boxtimes {n_1}^p \;=\; ({m_2}^{m_1} \boxtimes n_1)^p \;=\; \{(m_1,n_1),(m_2,n_2)\}^p$$

□

7.4 The crossed square associated to a 2-crossed module

Maybe there is no exact construction?

7.5 Homotopies of the actor 2-crossed module

[This needs significant revision.]

Definition 7.6 *A homotopy of the 2-crossed module $\mathcal{Z}$ is a pair of homomorphisms $\phi_\bullet = (\phi_1,\phi_0)$ such that $\phi_0 : R_\emptyset \to R_{\{1\}} \ltimes R_{\{2\}},\; p \mapsto (\phi_0^N p, \phi_0^M p)$ where*

$$\begin{aligned}
\phi_0(p_1p_2) &= (\phi_0^N(p_1p_2), \phi_0^M(p_1p_2)) \\
&= (\phi_0^N p_1, \phi_0^M p_1)(\phi_0^N p_2, \phi_0^M p_2) \\
&= (\phi_0^N p_1 \phi_0^N p_2, (\phi_0^M p_1)^{\phi_0^N p_2} \phi_0^M p_2)
\end{aligned}$$

So ϕ_0^M is a ϕ_0^N-derivation. $\phi_1 : R_{\{1\}} \ltimes R_{\{2\}} \to R_{[2]},\; (n,m) \mapsto (\phi_1^N n)(\phi_1^M m)$

$$\begin{aligned}
\phi_1(n,m) &= \phi_1((n,1)(1,m)) \\
&= \phi_1(n,1)\phi_1(1,m) \\
&= (\phi_1^N n)(\phi_1^M m)
\end{aligned}$$

[Moved what was Lemma 7.6 to cat2-group section.]

Lemma 7.7 *There is an action of $R_\emptyset \ltimes R_{\{2\}}$ on $R_{\{1\}} \ltimes R_{[2]}$ defined on the image of some derivation χ by*

$$\chi(p,m)^{(p_1,m_1)} = \chi(pp_1, m^{p_1}m_1)\chi(p_1,m_1)^{-1}$$

which is well defined **(should use the usual action!)**

Proof:

$$\begin{aligned}
\chi((p,m)^{(p_1,m_1)})^{(p_2,m_2)} &= (\chi(pp_1, m^{p_1}m_1)\chi(p_1,m_1)^{-1})^{(p_2,m_2)} \\
&= \chi(pp_1, m^{p_1}m_1)^{(p_2,m_2)}(\chi(p_1,m_1)^{(p_2,m_2)})^{-1} \\
&= \chi(pp_1p_2, m^{p_1p_2}m_1^{p_2}m_2)(\chi(p_2,m_2)^{-1}(\chi(p_1p_2, m_1^{p_2}m_2)\chi(p_2,m_2)^{-1})^{-1} \\
&= \chi(pp_1p_2, m^{p_1p_2}m_1^{p_2}m_2)\chi(p_1p_2, m_1^{p_2}m_2)^{-1} \\
\chi(p,m)^{(p_1,m_1)(p_2,m_2)} &= \chi(p,m)^{(p_1p_2, m_1^{p_2}m_2)} \\
&= \chi(pp_1p_2, m^{p_1p_2}m_1^{p_2}m_2)\chi(p_1p_2, m_1^{p_2}m_2)^{-1}
\end{aligned}$$

□

Lemma 7.8 *Consider a derivation* $\chi : R_{\emptyset} \ltimes R_{\{2\}} \to R_{\{1\}} \ltimes R_{[2]}$ *such that* $\chi(p,m) = (\chi^L(p,m), \chi^N(p,m))$. *Then the rules for* χ^L *and* χ^N *are as follows: ? ? ?*

Proof: we will show that above derivation is satisfies the derivation rule

$$\begin{aligned}
\chi((p_1,m_1)(p_2,m_2)) &= \chi(p_1p_2, m_1^{p_2}m_2) \\
\chi((p_1,m_1)(p_2,m_2)) &= \chi(p_1,m_1)^{(p_2,m_2)}\chi(p_2,m_2) \\
&= \chi(p_1p_2, m_1^{p_2}m_2)\chi(p_2,m_2)^{-1}\chi(p_2,m_2) \\
&= \chi(p_1p_2, m_1^{p_2}m_2)
\end{aligned}$$

□

Definition 7.9 *Now we can define a derivation which is depends on the derivation* χ^L

$$\begin{aligned}
\theta_p &= \chi^L(p,1) \\
\ddot{\chi}m &= \chi^L(1,m)
\end{aligned}$$

$$\begin{aligned}
\theta(p_1p_2) &= \chi^L(p_1p_2,1) \\
&= \chi^L((p_1,1),(p_1,1)) \\
&= (\chi^L(p_1,1))^{(p_2,1)}(\chi^L(p_2,1)) \\
&= (\theta p_1)^{p_2}(\theta p_2)
\end{aligned}$$

Lemma 7.10 $\chi^L : R_{\emptyset} \ltimes R_{\{2\}} \to R_{[2]}$ *is a derivation.*

Proof:

$$\begin{aligned}
\chi^L(p,m) &= \chi^L((p,1)(1,m)) \\
&= (\chi^L(p,1))^{(1,m)}(\chi^L(1,m)) \\
&= (\theta p)^m(\ddot{\chi}m)
\end{aligned}$$

□

8 Crossed n-cubes of groups

Here we include the basic ideas about crossed cubes (of groups) taken from Chapter 1 of Ellis' thesis [28], Ellis-Steiner [30], and Brown-Loday [17], and the associated cat^3-groups.

8.1 Crossed cubes

Here we include the basic ideas about crossed cubes (of groups) taken from Chapter 1 of Ellis' thesis [28], Ellis-Steiner [30], and Brown-Loday [17], and the associated cat^3-groups.
Let $[3] = \{1, 2, 3\}$, and let $A, B, C, \ldots$ be subsets of $[3]$.

Definition 8.1 *A crossed cube consists of the following.*

(i) *Groups R_A for each subset A of $[3]$, where we write R for $R_\emptyset$.*

(ii) *Group homomorphisms*

$$\partial_i : R_A \to R_{A\backslash\{i\}} \quad \textit{for all} \quad A \subseteq [3],\ i \in [3]$$

such that $\partial_i = \mathrm{id}_{R_A}$ when $i \notin A$, and $\partial_i \partial_j = \partial_j \partial_i$ for all $i, j \in [3]$.

Since the ∂_i commute, composite homomorphisms $\partial_B = \bigcirc_{i \in B} \partial_i : R_A \to R_{A\backslash B}$ are well defined and $\partial_A = \partial_{A\backslash B} \circ \partial_B$.

(iii) *For all $B \subseteq A$ an action of $R_{A\backslash B}$ on R_A making $\mathcal{R}_{A,A\backslash B} = (\partial_B : R_A \to R_{A\backslash B})$ a crossed module.*

For each $j \in [3]$ the maps

$$(1, \partial_j) : \mathcal{R}_{A,A\backslash\{i\}} \to \mathcal{R}_{A,A\backslash\{i,j\}} \quad \textit{and} \quad (\partial_j, 1) : \mathcal{R}_{A,A\backslash\{i,j\}} \to \mathcal{R}_{A\backslash\{j\},A\backslash\{i,j\}}$$

are crossed module homomorphisms.

It follows that all the actions act via R :

$$a^b \ = \ a^{\partial_B b} \quad \textit{for} \quad a \in R_A,\ b \in R_B, \quad \textit{and} \quad B \subseteq A.$$

(iv) *For all $A, B \subseteq [3]$ a crossed pairing $\boxtimes_{A,B} : R_A \times R_B \to R_{A \cup B}$, such that $(b \boxtimes_{B,A} a) = (a \boxtimes_{A,B} b)^{-1}$ and, when $B \subseteq A$, $\boxtimes_{A,B}$ is the principal crossed pairing for $\mathcal{R}_{A,B}$ given by $a \boxtimes b = a^{-1} a^b$ and $b \boxtimes a = (a^{-1})^b a$.*

(v) *Various axioms relating the homomorphisms, actions, and crossed pairings, for example*

- $\partial_i(a \boxtimes_{A,B} b) \ = \ (\partial_i a) \boxtimes_{A\backslash\{i\},B\backslash\{i\}} (\partial_i b).$
- **[More to follow?]**

Note that the $\boxtimes_{A,B}$ define actions of B on A for all $A, B \subseteq [3]$ by

$$a^b \ := \ a\ \partial_{B\backslash A}(a \boxtimes b).$$

8.2 Cat^3-groups

The corresponding notion of cat^3-group may be defined in a similar way.

Definition 8.2 *A cat^3-group consists of the following.*

(i) *Groups G_A for each subset A of $[3]$.*

(ii) *Group homomorphisms*

$$t_i, h_i : G_A \to G_{A\setminus\{i\}}, \quad e_i : G_{A\setminus\{i\}} \to G_A, \quad \textit{for all} \quad A \subseteq [3],\ i \in [3]$$

such that

- $t_i = h_i = e_i = \mathrm{id}_{R_A}$ *when* $i \notin A$,
- $t_i t_j = t_j t_i,\ h_i h_j = h_j h_i$ *and* $e_i e_j = e_j e_i$ *for* $i, j \in [3]$,
- $t_i h_j = h_j t_i$ *for* $i \neq j$.

Since the t_i, h_i and e_i commute, composite homomorphisms $t_B, h_B : R_A \to R_{A\setminus B}$ and $e_A : R_B \to R_{A\cup B}$ are well defined for all $A, B \subseteq [3]$.

8.3 Crossed n-cubes with $n \geqslant 4$

Let $[n] = \{1, 2, ..., n\}$, and let $A, B, C, \ldots$ be subsets of $[n]$.

A crossed n-cube consists of the following.

(i) Groups R_A for each subset A of $[n]$, where we write R for $R_\emptyset$.

(ii) Group homomorphisms

$$\partial_i : R_A \to R_{A\setminus\{i\}} \quad \text{for all} \quad A \subseteq [n],\ i \in [n]$$

such that $\partial_i = \mathrm{id}_{R_A}$ when $i \notin A$, and $\partial_i \partial_j = \partial_j \partial_i$ for $i, j \in [n]$.
Since the ∂_i commute, composite homomorphisms $\partial_B = \bigcirc_{i\in B} \partial_i : R_A \to R_{A\setminus B}$ are well defined and $\partial_A = \partial_{A\setminus B} \circ \partial_B$.

(iii) For all $B \subseteq A$ an action of $R_{A\setminus B}$ on R_A making $\mathcal{R}_{A,A\setminus B} = (\partial_B : R_A \to R_{A\setminus B})$ a crossed module.
For each $j \in [n]$ the maps

$$(1, \partial_j) : \mathcal{R}_{A,A\setminus\{i\}} \to \mathcal{R}_{A,A\setminus\{i,j\}} \quad \text{and} \quad (\partial_j, 1) : \mathcal{R}_{A,A\setminus\{i,j\}} \to \mathcal{R}_{A\setminus\{j\},A\setminus\{i,j\}}$$

are crossed module homomorphisms.
It follows that all the actions act via R :

$$a^b \;=\; a^{\partial_B b} \quad \text{for} \quad a \in R_A,\ b \in R_B, \quad \text{and} \quad B \subseteq A.$$

(iv) For all $A, B \subseteq [n]$ a crossed pairing $\boxtimes_{A,B} : R_A \times R_B \to R_{A\cup B}$, such that $(b \boxtimes_{B,A} a) = (a \boxtimes_{A,B} b)^{-1}$ and, when $B \subseteq A$, $\boxtimes_{A,B}$ is the principal crossed pairing for $\mathcal{R}_{A,B}$ given by $a \boxtimes b = a^{-1} a^b$ and $b \boxtimes a = (a^{-1})^b a$.

(v) Various axioms relating the homomorphisms, actions, and crossed pairings, for example

- $\partial_i(a \boxtimes_{A,B} b) = (\partial_i a) \boxtimes_{A\setminus\{i\},B\setminus\{i\}} (\partial_i b)$.
- when $i \in A \cap B$, so that $A \cup B = (A \setminus \{i\}) \cup B = A \cup (B \setminus \{i\})$,

$$a \boxtimes_{A,B} b \;=\; \partial_i a \boxtimes_{A\setminus\{i\},B} b \;=\; a \boxtimes_{A,B\setminus\{i\}} \partial_i b.$$

  (This means that we need only define $\boxtimes_{A,B}$ when $A \cup B = \emptyset$.)
- $(a \boxtimes b)^c \;=\; a^c \boxtimes b^c$ when $C \subseteq A$ and $C \subseteq B$. **[Is this correct?]**

What wish to define an *n-derivation*, which would seem to be a set of maps

$$\chi_{B,A} : R_B \to R_A \quad \text{for all} \quad B \subseteq A$$

satifying suitable axioms:

(i) perhaps $\chi_{B,A}(bb') = (\chi_{B,A}b)^{b'}(\chi_{B,A}b')$?

(ii) closure: $\chi_{B,A} \circ \chi_{C,B} = \chi_{C,A}$?

(iii) ???

Exercise 8.3 Derive the crossed square axioms from those of a crossed 2-cube.

[These are just some thoughts to be worked on!]

8.4 Cat^n-groups

A cat^n-group consists of the following.

(i) Groups G_A for each subset A of $[n]$.

(ii) Group homomorphisms

$$t_i, h_i : G_A \to G_{A\setminus\{i\}}, \quad e_i : G_{A\setminus\{i\}} \to G_A, \qquad \text{for all} \quad A \subseteq [n],\ i \in [n]$$

such that

- $t_i = h_i = e_i = \mathrm{id}_{R_A}$ when $i \notin A$,
- $t_i t_j = t_j t_i,\ h_i h_j = h_j h_i$ and $e_i e_j = e_j e_i$ for $i, j \in [n]$,
- $t_i h_j = h_j t_i$ for $i \neq j$.

Since the t_i, h_i and e_i commute, composite homomorphisms $t_B, h_B : R_A \to R_{A\setminus B}$ and $e_A : R_B \to R_{A\cup B}$ are well defined.

References

[1] M. Alp. GAP, crossed modules, cat1-groups: applications of computational group theory. Ph.D. thesis, University of Wales, Bangor (1997).

[2] M. Alp and C. D. Wensley. Enumeration of cat1-groups of low order. *Int. J. Algebra and Computation* **10** (2000) 407–424.

[3] Z. Arvasi and T. Porter. Simplicial and crossed resolutions of commutative algebras. *J. Algebra* **181** (1996) 426–448.

[4] R. Brown. Higher-dimensional group theory. In R. Brown and T. L. Thickstun, editors, *Low-dimensional topology*, volume 48 of *London Math. Soc. Lecture Note Series*, 215–238. Cambridge University Press (1982).

[5] R. Brown. From groups to groupoids: a brief survey. *Bull. London Math. Soc.* **19** (1987) 113–134.

[6] R. Brown. *Topology and groupoids*. BookSurge, LLC, Charleston, SC (2006). Third edition of *Elements of modern topology* [McGraw-Hill, New York, 1968; MR0227979], With 1 CD-ROM (Windows, Macintosh and UNIX).

[7] R. Brown and N. D. Gilbert. Algebraic models of 3-types and automorphism structures for crossed modules. *Proc. London Math. Soc. (3)* **59** (1989) 51–73.

[8] R. Brown and P. J. Higgins. On the connection between the second relative homotopy group and some related spaces. *Proc. London Math. Soc.* **36** (1978) 193–212.

[9] R. Brown and P. J. Higgins. On the algebra of cubes. *J. Pure Appl. Algebra* **21** (1981) 233–260.

[10] R. Brown and P. J. Higgins. Tensor products and homotopies for ω-groupoids and crossed complexes. *J. Pure Appl. Algebra* **47** (1987) 1–33.

[11] R. Brown and P. J. Higgins. Algebraic models of 3-types and automorphism structures for crossed modules. *Proc. London Math. Soc. (3)* **59** (1989) 51–73.

[12] R. Brown and P. J. Higgins. Crossed complexes and chain complexes with operators. *Math. Proc. Camb. Phil. Soc.* **107** (1990) 33–57.

[13] R. Brown and P. J. Higgins. The classifying space of a crossd complex. *Math. Proc. Camb. Phil. Soc.* **110** (1991) 95–120.

[14] R. Brown and J. Huebschmann. Identities among relations. In R. Brown and T. L. Thickstun, editors, *Low-dimensional topology*, volume 48 of *London Math. Soc. Lecture Note Series*, 153–202. Cambridge University Press (1982).

[15] R. Brown and I. İçen. Homotopies and automorphisms of crossed modules of groupoids. *Applied Categorical Structures* **11** (2003) 185–206.

[16] R. Brown, D. L. Johnson and E. F. Robertson. Some computations of non-abelian tensor products of groups. *J. Algebra* **111** (1987) 177–202.

[17] R. Brown and J.-L. Loday. Van Kampen theorems for diagram of spaces. *Topology* **26** (1987) 311–335.

[18] R. Brown and R. Sivera. *Nonabelian algebraic topology* (2004). A draft of Part 1 is available from `http://www.bangor.ac.uk/~mas010/nonab-a-t.html`.

[19] R. Brown and C. Spencer. $\mathcal{G}$-groupoids, crossed modules and the fundamental groupoid of a topological group. *Nede. Akad. Wetensch. Proc.* **79** (1976) 296–302.

[20] R. Brown and C. D. Wensley. On finite induced crossed modules, and the homotopy 2-type of mapping cones. *Theory and Applications of Categories* **1** (1995) 54–71.

[21] R. Brown and C. D. Wensley. Computing crossed modules induced by an inclusion of a normal subgroup, with applications to homotopy 2-types. *Theory and Applications of Categories* **2** (1996) 3–16.

[22] R. Brown and C. D. Wensley. Computation and homotopical applications of induced crossed modules. *J. Symbolic Computation* **35** (2003) 59–72.

[23] J. M. Casas and M. Ladra. Colimits in the crossed modules category in Lie algebras. *Georgian Math. J.* **7** (2000) 461–474.

[24] W. H. Cockroft. On two-dimensional aspherical complexes. *Proc. London Math. Soc. (3)* **4** (1954) 375–384.

[25] D. Conduché. Modules croisés généralisés de longeur 2. *J Pure Appl. Algebra* **34** (1984) 155–178.

[26] D. Conduché. Simplicial crossed modules and mapping cones. *Georgian Math J* **10** (2003) 623–636.

[27] P. J. Ehlers. Algebraic homotopy in simplicially enriched groupoids. Ph.D. thesis, University of Wales, Bangor (1993).

[28] G. Ellis. Crossed modules and their higher dimensional analogues. Ph.D. thesis, University of Wales, Bangor (1984).

[29] G. Ellis. Computing group resolutions. *J. Symbolic Comput.* **38 (3)** (2004) 1077–1118.

[30] G. Ellis and R. Steiner. Higher dimensional crossed modules and the homotopy groups of (n+1)-ads. *J. Pure and Appl. Algebra* **46** (1987) 117–136.

[31] M. S. et al. GAP–Groups, Algorithms, and Programming .

[32] M. Forrester-Barker. Representations of crossed modules and cat^1-groups. Ph.D. thesis, University of Wales, Bangor (2004).

[33] The GAP Group. *GAP – Groups, Algorithms, Programming, Version 4.4* (2004). (`http://www.gap-system.org`).

[34] N. D. Gilbert. Derivations, automorphisms and crossed modules. *Comm. in Algebra* **18** (1990) 2703–2734.

[35] Y. A. Gol'fand. On the automorphism group of the holomorph of a group. *Math. Sbornik.* **27** (1950) 333–350.

[36] D. Guin-Waléry and J.-L. Loday. Obstructions à l'excision en K-théorie algébraique. volume 854 of *Springer Lecture Notes in Math.*, 179–216 (1981).

[37] P. J. Higgins. *Categories and Groupoids*, volume 7 of *Reprints in Theory and Applications of Categories* (2005). Originally published by:Van Nostrand Reinhold, 1971.

[38] N. C. Hsu. The groups of automorphisms of the holomorph of a group. *Pacific. J. Math.* **11** (1961) 999–1012.

[39] N. C. Hsu. The holomorphs of free abelian groups of finite rank. *Amer. Math. Monthly* **72** (1965) 754–756.

[40] K. H. Kamps and T. Porter. 2-groupoid enrichments in homotopy theory and algebra. *K-Theory* **25** (2002) 373–409.

[41] J. L. Loday. Spaces with finitely many non-trivial homotopy groups. *J. App. Algebra* **24** (1982) 179–202.

[42] A. S. T. Lue. The centre of the outer automorphism group of a free group. *Bull. London. Math. Soc.* **11** (1979) 6–7.

[43] A. S.-T. Lue. Semi-complete crossed modules and holomorphs of groups. *Bull. London Math. Soc.* **11** (1979) 8–16.

[44] W. H. Mills. The automorphisms of the holomorph of a finite abelian group. *Trans. Amer. Math. Soc.* **85** (1957) 1–34.

[45] E. J. Moore. Graphs of Groups: Word Computations and Free Crossed Resolutions. Ph.D. thesis, University of Wales, Bangor (2001).

[46] A. Mutlu and T. Porter. Crossed squares and 2-crossed modules **02.03**.

[47] K. J. Norrie. Crossed modules and analogues of group theorems. Ph.D. thesis, King's College, University of London (1987).

[48] K. J. Norrie. Actions and automorphisms of crossed modules. *Bull. Soc. Math. France* **118** (1990) 129–146.

[49] L. Plotkin. *Groups of automorphisms of algebraic systems*. Wolters-Noordhoff, Groningen (1972).

[50] T. Porter. n-types of simplicial groups and crossed n-cubes. *Topology* **32** (1993) 5–24.

[51] W. Premans. Completeness of holomorphs. *Indag. Math.* **19** (1957) 608–619.

[52] J. H. C. Whitehead. On adding relations to homotopy groups. *Annals of Math.* **41** (1941) 806–810.

[53] J. H. C. Whitehead. Note on a previous paper entitled "On adding relations to homotopy groups". *Annals of Math.* **47** (1946) 806–810.

[54] J. H. C. Whitehead. On operators in relative homotopy groups. *Ann. of Math.* **49** (1948) 610–640.

[55] J. H. C. Whitehead. Combinatorial homotopy II. *Bull. Amer. Math. Soc.* **55** (1949) 453–496.